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Geometric Mean-Based Approximation Method for Discrete Chaotification in Chaotic Map

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How To Cite: Ashish. Geometric Mean-Based Approximation Method for Discrete Chaotification in Chaotic Map. *Applied Mathematics and Statistics* **2025**, *2*(1), 4. https://doi.org/10.53941/ams.2025.100004.

Received: 6 May 2025
Revised: 25 June 2025
Accepted: 26 June 2025
Published: 30 June 2025

Abstract: Dynamical systems are complex and constantly changing systems that exhibit predictable and unpredictable behaviour because of their inherent randomness and sensitivity to initial conditions. In the last few years, the dynamics of various fixed-point recursive methods and chaotic maps have received significant attention from the research community. Generally, in dynamical systems, the standard dynamics revolve around the chaotic map $\lambda p(1-p)$, where the growth rate parameter $\lambda \in [0,4]$. In this article, a novel Geometric Mean-Based fixed point recursive method is used to examine the dynamical behaviour in the chaotic map $\lambda p(1-p)$ in which the growth rate parameter $\lambda \in [0,4]$ approaches a maximum value of 6.7. Furthermore, the mathematical and computational study reveals the efficiency of the proposed approximation method. In this method, the logistic map admits extra freedom in the parameter λ, which gives improved dynamic properties such as fixed point, periodicity, chaos, and Lyapunov exponent. Additionally, it has been noted that better dynamic performance could enhance various applications such as weather forecasting, secure communications, neural networks, cryptography, and discrete traffic flow models, etc.

Keywords: Geometric Mean-Based Method; nonlinear dynamics; bifurcation; Lyapunov exponent

1. Introduction

The origins of nonlinear dynamical systems and chaos theory go back to Newton's three-body problem, which was acknowledged as unsolvable by Poincaré. The physical models around us are presented as nonlinear systems and might be used as quantitative tools to access the environmental behaviour in the real world. The crucial mathematical steps were taken by Lorenz [1] and May [2] in the computational work on nonlinear difference and differential equations. Afterwards, in the mid-19th century P. F. Verhulst introduced the nonlinear equation $rx(1-x)$, known as Logistic map, which played an eminent role in discrete nonlinear dynamical systems. In 1976, May [2] popularised this logistic equation as discrete nonlinear dynamical system. Furthermore, Feigenbaum [3] examined the dynamical properties of logistic equation analytically and experimentally. For most basic knowledge one may read Devaney [4,5], Alligood et al. [6], Ausloos and Dirickx [7], Andrecut [8], Holmgren [9], Block and Coppel [10], Diamond [11], Alphar [12].

Therefore, the discrete dynamical system plays a prominent role in chaos theory and they are applied in the modelling and in various scientific disciplines such as discrete traffic control systems, secure data communication systems, cryptography, neural networks, weather forecasting, etc. In 1987, Harikrishan et al. [13] established the period-doubling bifurcation structure in the logistic equation using control parameter r. Further, they examined that the structure can be bifurcated quantitatively and qualitatively after first bifurcation onward. Various generalisations of the one-dimensional logistic equation have been introduced by the researchers such as Radwan [14], Chowdhury et al. [15] and Sayed et al. [16]. Chaos, the utmost property in the dynamics have been used in the modelling of multiple methods of cryptography (see [17–19]). In 2010, a secure communication system to transfer the signals from one place to another was introduced with the help of chaotic signals by Singh and Meng [20]. A chaotic noise analog generator was designed using logistic map and MOSQT circuits by Medina et al. [21]. In 1996, Molina et al. [22] determined the time series embedding dimension for logistic and other discrete maps. In 2018, Ashish et al. [23] illustrated the dynamical properties of the logistic map, such as fixed-point, periodicity chaos, Lyapunov exponent using the Mann iterative procedure and also established the enhanced chaos based traffic control model. Further, in the next year, they introduced the novel fixed point approach which illustrates the dynamical behaviour in the standard logistic system [24]. For more study on the applications of discrete one-dimensional map one may read Radwan [14].

In 2023, Jinde Cao et al. [25] examined the scaling property using Euler's numerical method that is the most effective way to observe the transition to chaos. To observe the transition to chaos, various properties such as bifurcation scaling, fork width scaling, and Lyapunov exponent are illustrated. Recently, Sajid et al. [26] studied the stability in chaos through a hybrid control technique. Since the system depends on novel parameters κ, α, and r which make it unique from existing systems. In this method chaos is kicked with the parameter κ which stabilizes the chaos in different stationary states of order p. In recent years, M. A. Noor introduced various methods in different directions using innovative techniques to solve the complex problems. Moreover, Noor [27–29] introduced various higher order iterative approximation method which may help to determine the dynamical properties in nonlinear dynamical systems. Also, result the novel Noor approximation methods are helpful in several real-life applications such as chaos, Finance, climate change, design analysis, geometry and fractal analysis [30].

In this article, the various properties of nonlinear systems are studied using the Geometric Mean-Based approximation method. In the first section of the article, a brief introduction to the dynamical system is presented. Section 2, is the main body of the article, which has been divided into three subsections. The novel Geometric Mean-Based nonlinear dynamical system is introduced using a logistic map and Geometric mean-based fixed-point iterative procedure, followed by a fixed-point theorem and a remark. Further, the first subsection contains the time-series analysis, the second subsection determines periodicity evolution, and the third subsection examines the Lyapunov exponent, followed by Lyapunov exponent and counterexamples and suitable remarks. Finally, the complete conclusion and the future direction and applications in this direction are presented in Section 3.

2. Discrete Chaotification Using Geometric Approximation Method

An analytical study admits a major preamble to examine the dynamics of nonlinear systems using various fixed-point approximation methods. Here, the dynamical interpretation of the standard chaotic map using the Geometric Mean-Based fixed-point approximation Method is demonstrated. Let us consider the dynamical map

$$g(p) = \lambda p(1-p), \tag{1}$$

where the key parameter $\lambda \in [0,4]$ represents the growth-rate in the system and $p \in [0,1]$. For the starter p_0 in the closed interval [0, 1] implies p_1 as the new output. Then, by using the Geometric Mean-Based fixed-point approximation formula, we obtain

$$G_\lambda(p) = p_1 = \sqrt{p_0.g(p_0)}, \qquad \text{where} \qquad g(p_0) = \lambda p_0(1-p_0). \tag{2}$$

Inductively, we obtain the following dynamical system

$$p_n = \sqrt{p_{n-1}.g(p_{n-1})}, \qquad \text{where} \quad p_n \in [0,1] \quad and \quad n \in N. \tag{3}$$

The sequence p_n is known as the iterative orbit of the nonlinear system where all the dynamical activities are carried out by the system. Figure 1 presents the functional image of the given system $G_\lambda(p)$ with a critical value of 0.675 and the stable fixed point $1-\frac{1}{\lambda}$. The growth rate parameter λ will be determined in the further section which guarantees that $p_n \in [0,1]$. Figure 2 demonstrate the plot for $G^2_\lambda(p)$ versus the periodic components q_2^- and q_2^+ in blue colour. Moreover, it is observed that the novel superior dynamical system shows huge differences between the critical values, fixed and periodic states as compared to the values determined through simple one-dimensional chaotic map. Therefore, it is a matter of interest to examine the major results in chaotic map using Geometric Mean-Based fixed-point approximation method.

Remark 1. *It is observed that the critical value increases from 0.5 to 0.675 and the stable fixed point increases from 0.75 to 0.8507 for the utmost of the growth rate component λ in the superior Geometric dynamical system (3) as compared to standard chaotic map (1). A comparative analysis has been shown in Figure 1.*

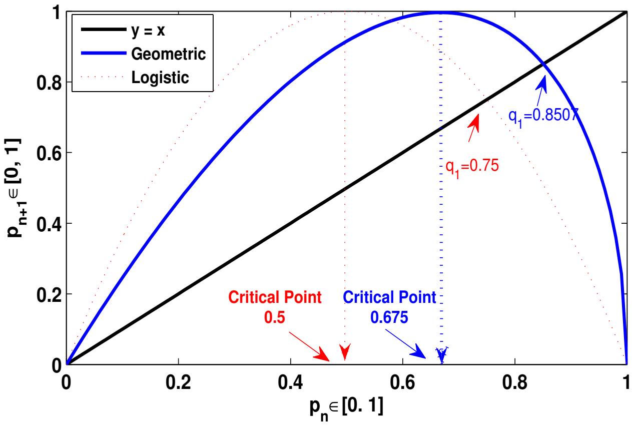

Figure 1. Dynamical image for system $G_\lambda(p)$ when $\lambda = 6.7$ and $p \in [0, 1]$.

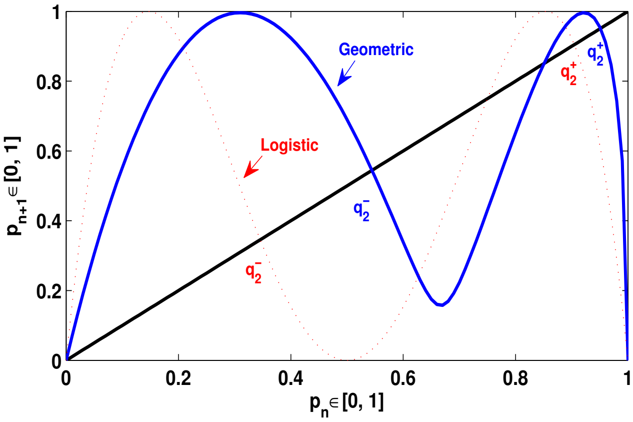

Figure 2. Dynamical image for system $G^2_\lambda(p)$ when $\lambda = 6.7$ and $p \in [0, 1]$.

Theorem 1. *Let $g(p)$ be the standard dynamical map and $G_\lambda(p)$ be the superior Geometric Mean-Based fixed-point approximation method, where $\lambda > 0$ and $p \in [0, 1]$. Then, determine 0 and $1 - \frac{1}{\lambda}$ as the two trivial fixed-point for the novel dynamical system $G_\lambda(p)$, where $p \in [0, 1]$ and $\lambda > 0$.*

Proof. Let, $g_\lambda(p)$ be the standard dynamical map and $G_\lambda(p)$ be the novel Geometric Mean-Based fixed-point approximation method. Then, from Equations (1) and (2), we get

$$G_\lambda(p) = \sqrt{p.\lambda p(1-p)}. \tag{4}$$

Now, to determine the fixed-point in the dynamical system (4), we take the definition of fixed point given by Devaney [4,5], then we get

$$\begin{aligned} \sqrt{p.\lambda p(1-p)} &= p \\ p.\lambda p(1-p) &= p^2 \\ p.\lambda p(1-p) - p^2 &= 0 \\ p^2(\lambda - p\lambda - 1) &= 0 \end{aligned} \tag{5}$$

Then, solving Equation (5), we determine the fixed point $p = 0$ and $p = 1 - \frac{1}{\lambda}$. The fixed point $1 - \frac{1}{\lambda}$ is the stable fixed state which depends on the growth rate parameter λ. Figure 1 shows that the for $\lambda = 6.7$ the stable state approaches to 0.8507. This completes the proof. In further sections the experimental study is carried out followed by the time-series analysis, periodic analysis and Lyapunov exponent. □

2.1. Time Series Analysis

The growth rate parameter λ, a major entity in one-dimensional equations always affects the dynamical behaviour of a nonlinear system in chaos theory. In this part, the dynamics of a nonlinear logistic map is examined using Geometric Mean-Based fixed-point iterative method and the maximum of the growth rate parameter λ is determined. It is observed that the superior Geometric system $G_\lambda(p)$ shows a major increment in the range of the growth rate parameter λ as compared to the standard logistic system $\lambda p(1-p)$. Figures 3–8 present the time-series study of the fixed, periodicity and chaotic states of the superior Geometric system $G_\lambda(p)$.

Figures 3–8 show that the superior Geometric system $G_\lambda(p)$ admits all the dynamical propertied like fixed, periodicity, and chaos for the growth rate parameter $0 \leq \lambda \leq 6.7$. Figure 3, shows that the iterative sequence $\{p_n\}$ approaches the fixed point $1-\frac{1}{\lambda}$ for the growth rate value $1 \leq \lambda \leq 5$. When we take $\lambda = 4$, it admits the stable fixed point $q^* = 0.75$. Further, it observed that as the value of λ approaches through 5, the system starts to introduce the orbits of periodicity 2^n, $n \in N$. For $5 < \lambda \leq 5.855$, the system oscillates between periodicity of order-2. Figure 4 shows that at $\lambda = 5.5$ the trajectory vibrates between $q_2^- = 0.6607$ and $q_2^+ = 0.9026$. As the parameter λ approaches through 5.855 the iterative orbit $\{p_n\}$ starts to vibrate into period-4 cycle for the growth rate parameter range $5.855 < \lambda \leq 6.036$. As shown in Figure 6, at $\lambda = 6$ the bifurcation trajectory vibrate between $q_{41}^- = 0.5544$, $q_{41}^+ = 0.6790$, $p_{42}^- = 0.9065$ and $p_{42}^+ = 0.9423$. When the growth rate parameter λ crosses the limit beyond 6.036, the behaviour of the orbit $\{p_n\}$ becomes more complicated. Figure 7 shows that the orbit of the system $G_\lambda^3(p)$ fluctuate in periodicity of order 8 for $6.036 < \lambda \leq 6.069$.

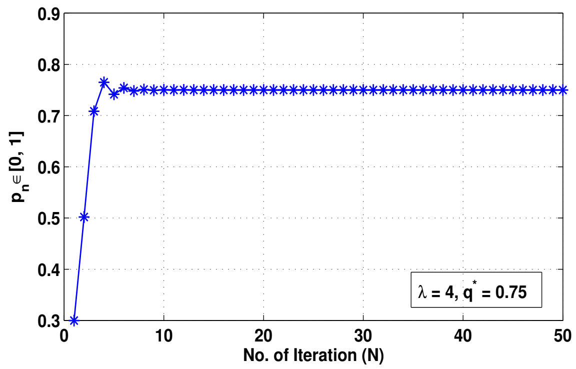

Figure 3. Fixed-point stable plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 4$.

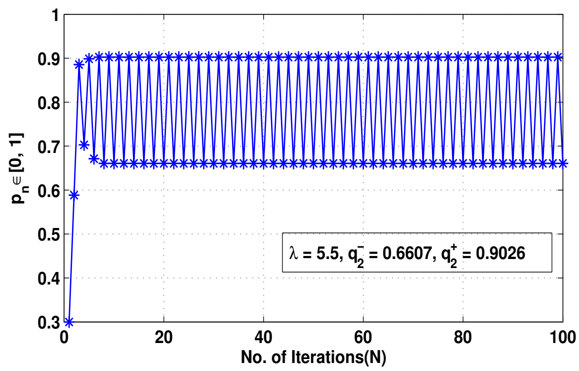

Figure 4. Period-2 stable plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 5.5$.

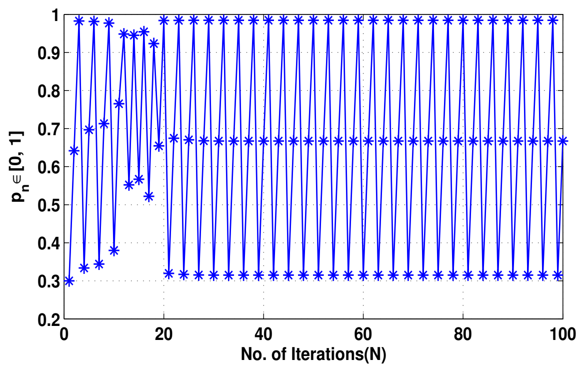

Figure 5. Period-3 stable plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 6.54$.

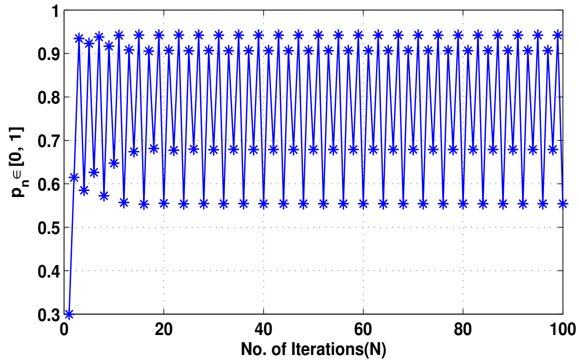

Figure 6. Period-4 stable plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 6$.

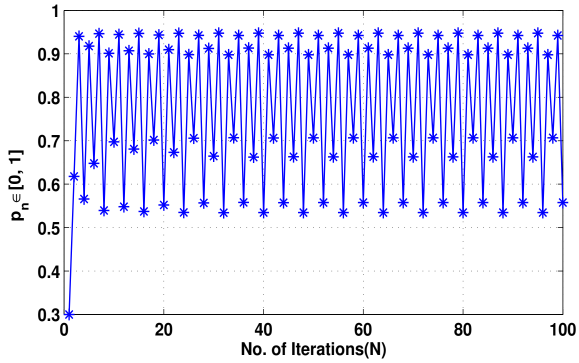

Figure 7. Period-8 stable plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 6.06$

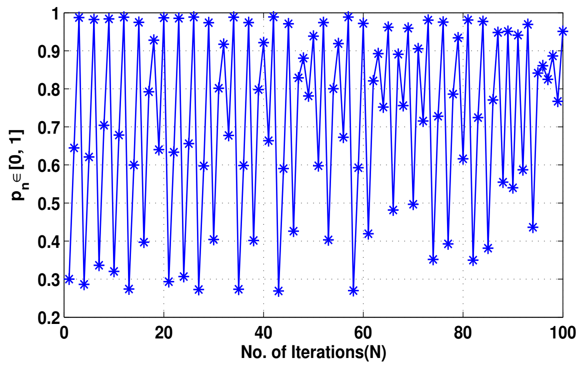

Figure 8. Aperiodicity plot for $G_\lambda(p)$ when $p_0 = 0.3$ and $\lambda = 6.6$.

Proceeding in this way, when $\lambda \approx 6.07$ the system $G_\lambda(p)$ becomes chaotic. Figure 8 represents the complete irregular orbit behaviour and the Figure 5 shows the representation of period-3 window when λ lies between 6.53 to 6.56. For $\lambda = 6.54$, the trajectory vibrates in the periodic cycle order-3 between 0.3152, 0.6670 and 0.9843.

2.2. Periodicity Analysis

Period-doubling an another important feature in the dynamics of nonlinear one dimensional systems which is used to determine the evolution from regularity to chaos. It shows all the periodic and chaotic behaviour admitted by the nonlinear system for different orders of periodicity and chaos. Therefore, in this section, we illustrate the dynamics of logistic map using Geometric Mean-Based fixed-point recursive method $G_\lambda(p)$.

For the starter $p_0 \in [0, 1]$ with stepsize $h = 0.001$, the period-doubling analysis is carried out as shown in the given Figures 9–12. Here, the question arises, "What is the next target when the growth rate parameter λ approaches through a breakout of growth rate parameter $\lambda = 5$?". This is the place where the given novel Geometric system illustrates most powerful properties. The complete bifurcation diagram is shown in Figure 9 which shows the complete dynamics of the logistic map in Geometric Mean-Based fixed-point recursive method $G_\lambda(p)$. For $\lambda > 5$ the dynamical system vibrates between the two fixed points q_2^- and q_2^+. Figure 10 shows this behaviour in the period-doubling diagram, where the stationary state $1 - \frac{1}{\lambda}$ approaches to its maximum at $\lambda = 5$ and then bifurcates into two spikes. Such type of periodicity is known as period-2 cycle and lies in the growth rate parameter range $5 < \lambda \leq 5.855$. Again, as the parameter λ approaches through 5.855, the system bifurcates into periodicity of order 4. For the range $5.855 < \lambda \leq 6.036$ the system bifurcates into q_{41}^{-}, q_{41}^{+}, q_{42}^{-}, and q_{42}^{+} as shown in the Magnified Figure 10. Similarly, the periodicity continues for the higher orders of order 2^n as λ approaches to 6.069. Figure 10 gives the complete periodicity behaviour for the order 2^n.

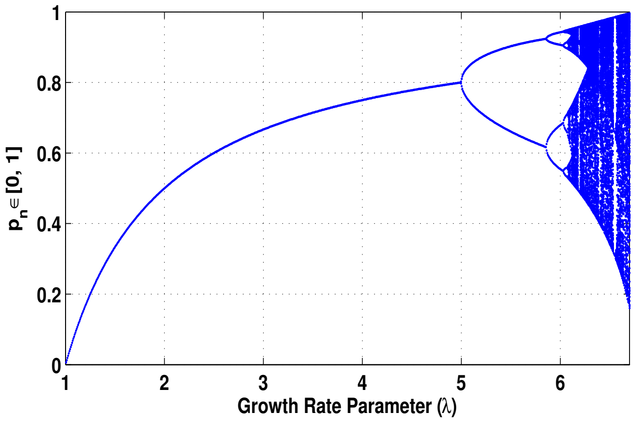

Figure 9. Period-doubling plot for $G_\lambda(p)$ when $0 \leq \lambda \leq 6.7$.

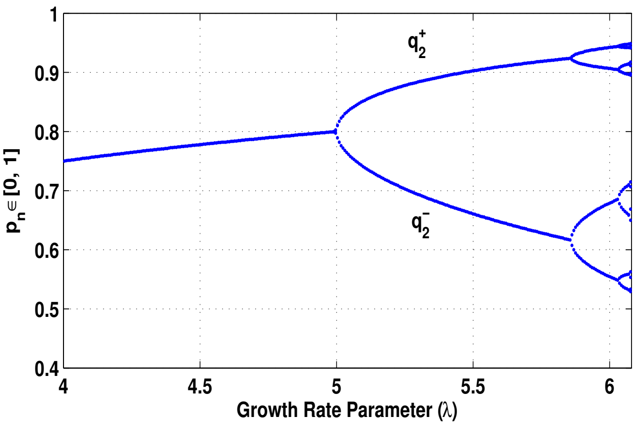

Figure 10. Periodicity plot for $G_\lambda(p)$ when $1 \leq \lambda \leq 6.069$.

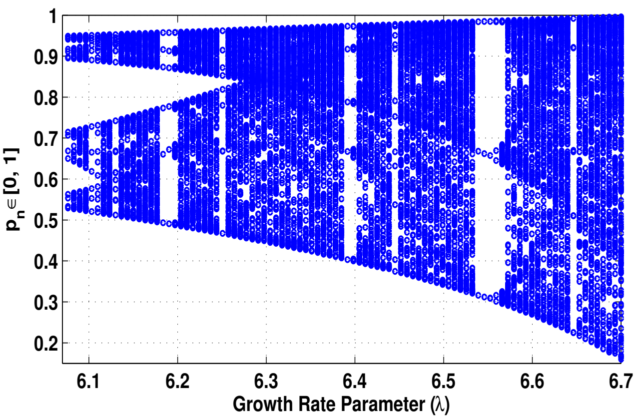

Figure 11. Chaotic regime plot for $G_\lambda(p)$ when $6.07 \leq \lambda \leq 6.7$.

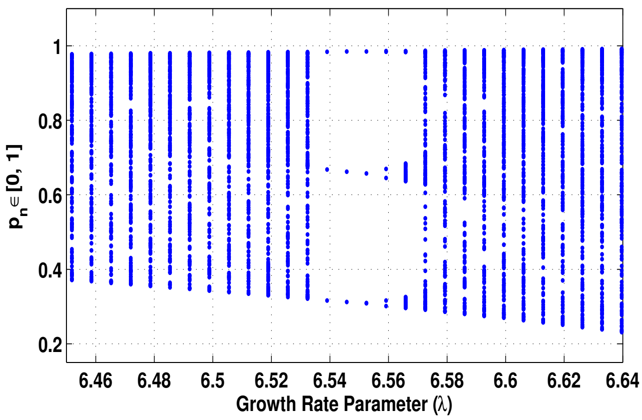

Figure 12. Magnified Chaotic regime plot for $G_\lambda(p)$ when $6.4 \leq \lambda \leq 6.64$.

Amazingly, it is the most sensitive situation, that is, $\lambda_\infty \approx 6.07$, where the system admits bifurcation route to chaos. As approaches beyond $\lambda = 6.07$, the system gets fully chaotic as shown in the magnified region in Figure 11. When we zoom Figure 9 in the range $6.07 < \lambda \leq 6.7$ the beauty of chaos is determined as shown in Figure 11. We can see that in magnified regime there are also many small windows of other periodic orders. But the most important window in the chaotic regime is period-3 window. It is important because according to Sarkovaski theorem it is the window that confirms the chaos and periodicity of other orders in a nonlinear dynamical system. Figure 12 shows the magnified version of the period-3 window for the growth rate parameter $6.53 \leq \lambda \leq 6.56$.

Remark 2. *From the period-doubling diagram it is examined that the periodic regime and the chaotic regime has the larger range of the growth rate parameter λ as shown in the Figures 10 and 11 as compared to regime described in the standard logistic system.*

2.3. Lyapunov Exponent Analysis

In this part, we illustrate the Lyapunov exponent property of the Logistic map using Geometric Mean-Based fixed-point iterative method $G_\lambda(p)$ which determines the sensitivity behaviour in the system depending on initial conditions. The positive Lyapunov value determines the irregularity and negative Lyapunov value shows the stability in the dynamical system. For each value of the growth rate parameter $1 \leq \lambda \leq 8$ the Lyapunov value is determined as shown in Figure 13. Therefore, for the logistic system $g(p)$ in the given Geometric Mean-Based fixed-point iterative method $G_\lambda(p)$, the Lyapunov exponent is illustrated in the following way:

Let p and $p + \varepsilon$ be the initial points, where $0 < \varepsilon < 1$ is infinite small separation. The difference between two trajectories is represented by $\Delta = \varepsilon.e^{n\eta}$, where η is the Lyapunov exponent. The Δ is taken as an exponential growth. Inductively, we have

$$\begin{aligned}
& & G_\lambda^n(p+\varepsilon) - G_\lambda^n(p) &= \Delta, \\
&\text{i.e.} & G_\lambda^n(p+\varepsilon) - G_\lambda^n(p) &= \varepsilon.e^{n\eta}, \\
&\therefore & \frac{G_\lambda^n(p+\varepsilon) - G_\lambda^n(p)}{\varepsilon} &= e^{n\eta}, \\
& & \lim_{\varepsilon\to\infty} \frac{G_\lambda^n(p+\varepsilon) - G_\lambda^n(p)}{\varepsilon} &= e^{n\eta}, \\
&\text{i.e.} & (G_\lambda^n)'(p) &= e^{n\eta}. \qquad (6)
\end{aligned}$$

Now, applying the Logarithm throughout the Equation (6), then, we get

$$\eta = \frac{1}{n}\log|(G_\lambda^n)'(p)|, \qquad (7)$$

where λ is the growth rate parameter and $(G_\lambda^n)'(p)$ represents the derivative for the Geometric system $G_\lambda^n(p)$. But the differentiation of $G_\lambda^n(p)$ is determined by chain rule method. Thus, we get

$$(G_\lambda^n)'(p_1) = G_\lambda'(p_n) \times G_\lambda'(p_{n-1}) \times \ldots G_\lambda'(p_2) \times G_\lambda'(p_1). \qquad (8)$$

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Then, taking (7) and (8), we find

$$\eta = \frac{1}{n}\log|G'_\lambda(p_n) \times G'_\lambda(p_{n-1}) \times \ldots G'_\lambda(p_2) \times G'_\lambda(p_1)|,$$
$$\eta = \frac{1}{n}[\log|G'_\lambda(p_n)| + \log|G'_\lambda(p_{n-1})| + \ldots + \log|G'_\lambda(p_2)| + \log|G'_\lambda(p_1)|],$$
$$\eta = \frac{1}{n}\sum_{i=1}^{n}\log|G'_\lambda(p_i)|, \tag{9}$$

When the trajectory of the system approaches to a fixed point, then from Equation (9), we get

$$\eta = \log|G'_\lambda(p_1)|. \tag{10}$$

Also, as the iterative trajectory vibrates in periodicity of order q then we get

$$\eta = \frac{1}{q}\sum_{i=1}^{q}\log|G'_\lambda(p_i)|. \tag{11}$$

To examine the Lyapunov exponent for the irregular orbits we can take complete length of the iterative orbit. But for irregular orbits it is impossible to take full length of iterative orbit. Therefore, the finite number of terms are taken to examine the Lyapunov exponent.

Remark 3. *Finally, it is noticed that for the parameter $\eta > 0$ the system admits chaotic behaviour and for $\eta < 0$ the system approaches to stable fixed and periodic states as shown in Figure 13.*

Example 1. *Let $G_\lambda(p)$ be the superior Geometric Mean-Based fixed-point system and $g(p) = \lambda p(1-p)$ be the logistic map. Then, find out the Lyapunov exponent for the system $G_\lambda(p)$, $\lambda \in [0, 6.7]$ and $p \in [0, 1]$ for (i) the fixed point at $\lambda = 4$ and (ii) the periodic points at $\lambda = 5.5$.*

Solution. (i) We know that for $1 < \lambda \leq 5$ the superior Geometric Mean-Based system $G_\lambda(p)$ admits the stable fixed point $1 - \frac{1}{\lambda}$. Therefore, when we take $\lambda = 4$ then it gives the fixed point $1 - \frac{1}{\lambda} = 0.75$. Then, using Equation (10) we get the required Lyapunov exponent. Then from Equation (2), we get

$$G_\lambda(p) = \sqrt{p.\lambda p(1-p)} \tag{12}$$

Taking the derivation on both sides, we obtain

$$G'_\lambda(p) = \frac{2\lambda - 3\lambda p}{2\sqrt{\lambda(1-p)}} \tag{13}$$

Putting $\lambda = 4$ and $p = 0.75$, then we get

$$G'_4(0.75) = \frac{-1}{2} = -0.5 \tag{14}$$

Then, from Equations (10) and (14), we can write

$$\eta = \log|-0.5| = -0.3010$$

Hence, $\eta = -0.3010$ is the Lyapunov exponent for $\lambda = 4$ and the fixed point 0.75. The obtained Lyapunov value is negative which shows that the given fixed point is stable.

(ii) We know that for $5 < \lambda \leq 5.855$ superior Geometric Mean-Based system $G_\lambda(p)$ exhibits the stable periodic cycle of order 2. Therefore, $\lambda = 5.5$, let us take $q_2^- = 0.6607$ and $q_2^+ = 0.9026$. Then from Equation (11) we get

$$G'_{5.5}(0.6607) = \frac{0.0984}{1.1648} = 0.0844 \tag{15}$$

and

$$G'_{5.5}(0.9026) = \frac{-3.8929}{0.6241} = -6.2376 \tag{16}$$

Then, from (11), (15) and (16), we can say

$$\eta = \frac{1}{2}[\log|0.0844| + \log|-6.2376|] = -0.139$$

Thus, $\eta = -0.139$ is the Lyapunov exponent at $\lambda = 5.5$. The obtained Lyapunov exponent is negative which shows that the given periodic point is stable.

In above part, we have studied the method to determine the maximum Lyapunov exponent η in Harmonic Mean-Based fixed-point recursive method $G_\lambda(p)$. The Laypunov exponent is illustrated for $\lambda \in [0, 6.7]$ and the maximum Lyapunov is determined as 0.6077. Then, the individual Lyapunov is also calculated for fixed and periodic state of the system as shown in Example 1. From the bifurcation diagrams it is clear that the maximum growth rate value is approaches to 6.7. Figures 13–16 shows the complete Lyapunov exponent behaviour for Geometric Mean-Based fixed point iterative system $G_\lambda(p)$. Figure 13 shows that for $0 \leq \lambda \leq 6.07$ it gives the negative Lyapunov exponent in the periodicity of order 2^n exists and for $6.07 < \lambda \leq 6.7$ it give positive Lyapunov value in the which the chaotic behaviour of the system exists. See Magnified Figure 14 in which all the spikes exist in negative region. The smallest order periodicity gives lowest Lyapunov exponent while as the order of periodicity increases the Lyapunov exponent also increases. While Figure 15 shows the magnified version the positive Lyapunov exponent for $6.07 < \lambda \leq 6.7$. In this diagram, it is clear that there also exist many periodic windows in the chaotic regime because many lobes are entering the negative regime. For $6.53 \leq \lambda \leq 6.56$, the period-3 window is seen that have negative Lyapunov exponent in the chaotic regime. This window is further zoomed in Figure 16. In Table 1, we have presented the Lyapunov exponent for the some selected values of the growth rate parameter $0 \leq \lambda \leq 6.7$. Further, to examine the efficient behaviour in the dynamics of Logistic Geometric Mean-Based System, a comparative analysis is also presented in Figure 17. Figure 17 shows the comparative Lyapunov exponent analysis versus standard Logistic system and Logistic Geometric Mean-Based method. The maximum LE in the standard Logistic system is 0.6932 using Picard orbit while the maximum LE using novel Geometric Mean-Based system is 0.6077. Further, Figure 18 shows that as the parameter λ increases through 6.07 the Lyapunov exponent increases sharply except the dips created by the periodic windows in the chaotic regime.

Remark 4. *It is analysed that the in the dynamics of logistic map using Geometric Mean-Based fixed-point recursive system there exists a number of periodic windows in the chaotic regime in which the Lyapunov lobes approach to negative value as shown in the magnified Figure 15.*

Remark 5. *Further, it is observed that in the superior Geometric Mean-Based system the periodicity and the chaotic regime exist for the larger range of the parameter λ as compared to standard logistic system.*

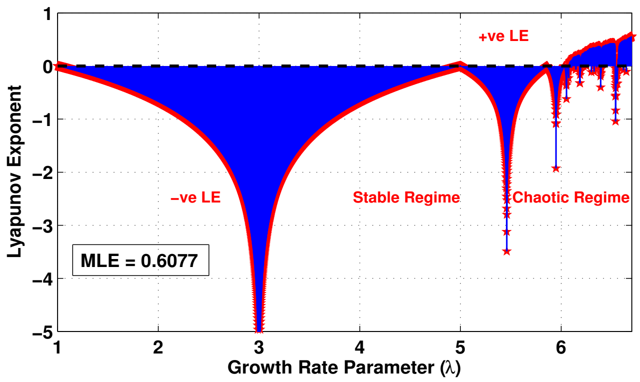

Figure 13. Lyapunov exponent plot for the system $G_\lambda(p)$ for $0 \leq \lambda \leq 6.7$.

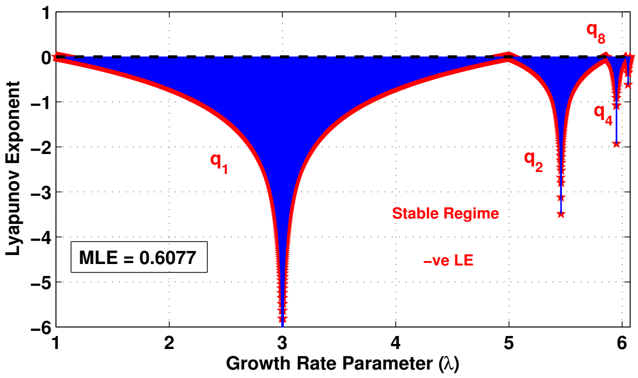

Figure 14. Periodic Lyapunov exponent plot for the system $G_\lambda(p)$ for $1 \le \lambda \le 6.07$.

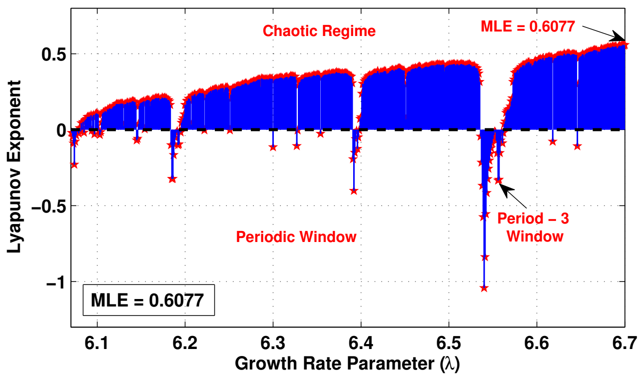

Figure 15. Chaotic Lyapunov exponent plot for the system $G_\lambda(p)$ for $6.07 \le \lambda \le 6.7$.

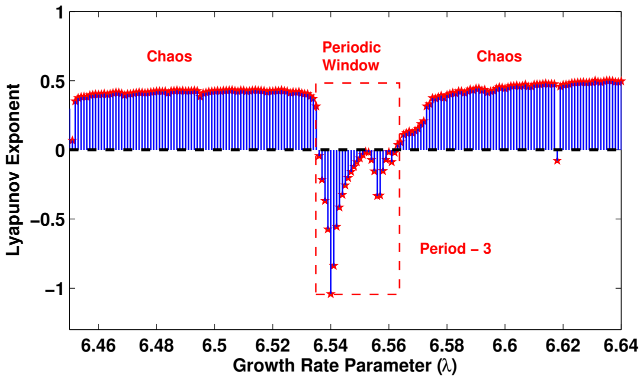

Figure 16. Magnified Period-3 window Lyapunov exponent plot for the system $G_\lambda(p)$.

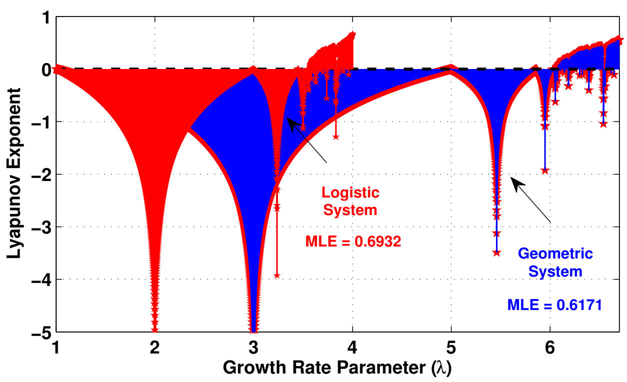

Figure 17. Comparative Lyapunov exponent versus Logistic and Harmonic system.

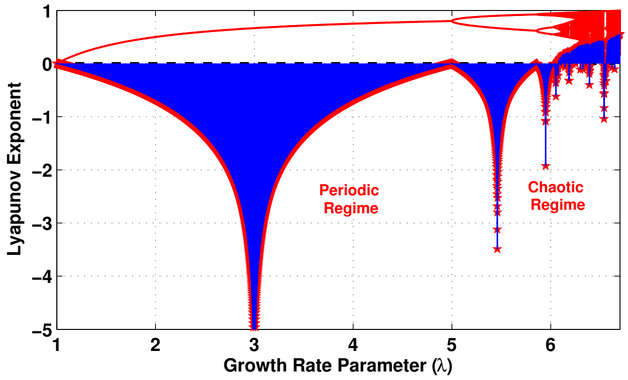

Figure 18. Comparative plot versus Lyapunov exponent spectrum and bifurcation.

3. Conclusions

In this article, by using the Geometric Mean-Based fixed-point approximation method, the dynamics in the logistic map are studied. The study summarises all the dynamical properties like fixed-point, periodicity, chaos and maximum Lyapunov exponent. Further, it is observed that as compared to standard logistic map in the Picard method the dynamics of the logistic map in the Geometric Mean-Based fixed-point approximation method perform superior dynamical properties for the larger range of the growth rate parameter λ.

In the second part, a few analytical results are discussed using functional diagrams and it is observed that the critical value of the system increases from 0.5 to 0.675 and the stable fixed point $1 - \frac{1}{\lambda}$ increases from 0.75 to 0.8507 for the full range of growth rate parameter $0 \leq \lambda \leq 6.7$. In the third part, the time-series analysis is illustrated and the behaviour of stable fixed point, periodic points of order 2, 3, 4, and 8, and chaos is studied. Period-doubling analysis is established in part four using bifurcation diagrams. The study contains magnified images of the periodicity, chaos and period-3 window region. In part four, the analytical analysis of maximum Lyapunov exponent is described followed by Example and Remarks. Figures 13–18 give the complete experimental analysis with a comparative analysis versus Lyapunov exponent in Picard method and Geometric Mean-based fixed point method. The maximum Lyapunov in Picard orbit is 0.6932 and in Geometric Mean-based fixed point method is 0.6077.

Finally, it is concluded that due to the higher range of growth rate parameter in periodic and chaotic region may improve the efficiency in chaos based applications such as traffic control system, cryptography, security systems, etc.

Author Contributions

The author has read and agreed to the published version of the manuscript.

Funding

This research received no external funding.

Institutional Review Board Statement

Not applicable.

Informed Consent Statement

Not applicable.

Data Availability Statement

Not applicable.

Conflicts of Interest

The author declares no conflict of interest.

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