# Independent Domination in Claw-Free Cubic Graphs

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**Abstract:** A vertex set $S$ of a graph $G$ is called an independent dominating set if $S$ is an independent set and each vertex in $V(G) \setminus S$ is adjacent to a vertex in $S$. The independent domination number $i(G)$ of $G$ is the minimum cardinality of an independent dominating set in $G$. This paper first proves that if $G$ is a connected $K_{1,3}$-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$. Meanwhile, $i(G) = \frac{1}{3}|V(G)|$ if and only if $G \in \mathcal{H}$, where $\mathcal{H}$ is an infinite cubic family with each graph being a $C_6^+$-necklace. Then, it is shown that if $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $C_3 \square K_2$-component, then $i(G) \leq \frac{5}{18}|V(G)|$. This result is tight.

**Keywords:** independent domination; claw-free; cubic graphs

## 1. Introduction

Let $G = (V, E)$ be a graph. A set $S$ of vertices in a graph $G$ is called an *independent dominating set*, abbreviated as ID-set, if $S$ is an independent set and every vertex in $V(G) \setminus S$ is adjacent to a vertex in $S$. The *independent domination number* of $G$, denoted as $i(G)$, is the minimum cardinality of an ID-set, and an ID-set of cardinality $i(G)$ in $G$ is called an $i(G)$-set. For recent books on independent domination, please refer to [1, 2].

The notations and graph theory terminologies in this paper generally follow [3]. The *degree* of a vertex in graph $G$ is denoted as $d_G(v)$, abbreviated as $d(v)$. For an integer $k \geq 1$, a $k^+$-vertex is a vertex having a degree of at least $k$. The maximum degree among the vertices of $G$ is denoted as $\Delta(G)$. The *open neighborhood* $N_G(v)$ of a vertex $v$ in $G$ is the set of neighbors of $v$, while the *closed neighborhood* of $v$ is the set $N_G[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, the subgraph induced by $S$ is denoted as $G[S]$, and $G$-$S$ is abbreviate as $G[V(G) \setminus S]$. Let $[k]$ be the set $\{1, 2, \ldots, k\}$ for a positive integer $k$. A cycle on $n$ vertices is denoted as $C_n$. For vertex $v \in V(G)$, let $u$ be a neighbor of $v$ and $u$ be on a triangle $T$. Then, $v$ is *adjacent* to triangle $T$, and $u$ is *incident* with $T$.

A graph is *F-free* if it does not include $F$ as an induced subgraph. A *claw* is a star $K_{1,3}$. A *diamond* is a $K_4 - e$, where $e$ is referred to as the missing edge. In this paper, $K_4 - e$ is abbreviated as $K_4^-$. $C_6^+$ is defined for the simple graph obtained from two vertex disjoint triangles by adding two vertex disjoint edges to it. In Figure 1, from left to the right, the first three subgraphs are $K_{1,3}$, $K_4^-$, and $C_6^+$, respectively. A *k-regular* graph is a graph where every vertex has a degree of $k$. If $k = 3$, then the graph is a *cubic* graph. Independent domination in cubic graphs and claw-free graphs has been extensively investigated in the literature (e.g., [3–8], etc.).

For a connected $k$-regular graph $G$ where $k \geq 1$. Rosenfeld [9] pointed out that $i(G) \leq \frac{V(G)}{2}$, and this is tight only for the balanced complete bipartite graph $K_{k,k}$. For a cubic graph, Lam, Shiu, and Sun [10] established the following upper bound on the independent domination number.

**Theorem 1.** *[10] If $G$ is a connected cubic graph of order $n$, where $n \geq 8$, then $i(G) \leq 2n/5$, and the bound is tight by $C_5 \square K_2$. See Figure 1d.*

Goddard and Henning [3] speculated that there is only one graph where the upper bound in Theorem 1 is tight. Dorbec et al. [11] claimed that the conjecture holds if, in addition, $G$ does not have a subgraph isomorphic to $K_{2,3}$.

**Theorem 2.** *[11] If $G \neq C_5 \square K_2$ is a connected cubic graph of order $n$ that does not have a subgraph isomorphic to $K_{2,3}$, then $i(G) \leq 3n/8$.*
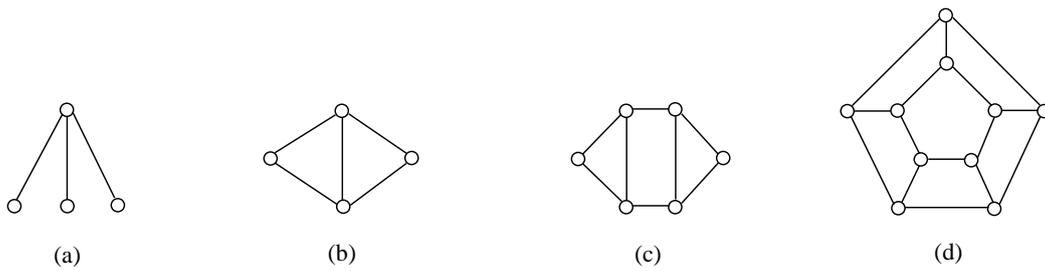


(a)  (b)  (c)  (d)

**Figure 1.** Three subgraphs and $C_5 \square K_2$.

The number of vertices in the largest independent set of a graph $G$ is referred to as the independent number and is denoted as $\beta(G)$. Murugesan et al. [12] pointed out that an independent set of a graph $G$ is dominating if and only if it is maximal. Thus, $\beta(G)$ is a natural upper bound of $G$ on the independent dominating number. Wang [13] obtianed the exact values for $\{K_{1,3}, K_4^-\}$-free cubic graphs on the independent number.

**Theorem 3.** *[13] For every connected claw-free cubic graph $G$ of order $n$, if $G$ ($G \neq K_4$) includes no $K_4^-$ as an induced subgraph, then $\beta(G) = n/3$.*

Therefore, if $G$ is a $\{K_{1,3}, K_4^-\}$-free cubic graph, then $i(G) \leq \frac{|V(G)|}{3}$. In this paper, it is shown that if $G$ is a connected $K_{1,3}$-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$, and this bound is tight. Furthermore, it is proven that if $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \square K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$, and this bound is best possible.

## 2. Main Results

A graph $G$ is said to be *subcubic* if its maximum degree is three. Let $n_j(G)$ be the number of vertices of degree $j$ in $G$. For a graph $H$, an $H$-*unit* in $G$ is called an induced subgraph of $G$ that is isomorphic to $H$. An edge $e$ is called a *triangle edge* if it is on a triangle, and otherwise, $e$ is a *flat edge*.

**Definition 1.** *A graph $G$ is SP-1 if $G = K_3$ or the following three properties hold: (i) $G$ is connected, (ii) $\Delta(G) = 3$, and (iii) every vertex belongs to a triangle.*

**Theorem 4.** *If $G$ is an SP-1 graph, then $3i(G) \leq n_2(G) + n_3(G)$.*

An infinite family $\mathcal{H}$ with an independent domination number $\frac{1}{3}$ of its order can be established as follows. Given $k$ disjoint copies $F_1, F_2, \ldots, F_k$ of $C_6^+$, where $V(F_i) = \{o_i, f_i, g_i, h_i, p_i, q_i\}$, and $o_i f_i g_i h_i p_i q_i o_i$ is a 6-cycle with two chords $f_i q_i$ and $g_i p_i$. Let $H_k$ be acquired from the disjoint union of these $k$ $C_6^+$ by adding the edges $\{h_i o_{i+1} : i \in [k-1]\}$ and $h_k o_1$. When $k = 1$, $H_1 = C_3 \square K_2$. Let $\mathcal{H} = \{H_k : k \geq 1 \text{ be an integer}\}$.

**Theorem 5.** *If $G$ is a connected claw-free cubic graph, then $i(G) \leq \frac{1}{3}|V(G)|$. Meanwhile, $i(G) = \frac{1}{3}|V(G)|$ if and only if $G \in \mathcal{H}$.*

For every $C_6^+$-unit $F_i$ in $H_k$, $i \in [k]$, $V(F_i)$ contains at least two vertices in an ID-set of $H_k$. Therefore, $i(H_k) \geq \frac{1}{3}|V(H_k)|$. Let $S = \bigcup_{i=1}^k \{f_i, p_i\}$, then $S$ is an ID-set in $H_k$. So, $i(H_k) = \frac{1}{3}|V(H_k)|$. For instance, when $k = 4$, a minimum ID-set of $H_4$ is illustrated in Figure 2.
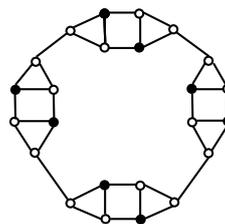


**Figure 2.** An $i(H_4)$-set of $H_4$ indicated with darkened vertices.

**Definition 2.** *Let $G$ be a graph, with $G_1, G_2, \ldots, G_k$ being the components of $G$. A graph $G$ is SP-2 if the following three properties hold for each $i \in [k]$: (i) $\Delta(G_i) = 3$, (ii) every vertex belongs to a triangle in $G_i$, and (iii) $G_i$ has no $K_4^-$-unit or $C_6^+$-unit.*

**Theorem 6.** *If $G$ is an SP-2 graph without $(C_3 \square K_2)$-component, then $18i(G) \leq 8n_2(G) + 5n_3(G)$.*

**Corollary 1.** *If $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph without $(C_3 \square K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$.*

This bound is tight, and Figure 3 shows the darkened vertices form an $i(G)$-set of $G$ of cardinality $\frac{5}{18}|V(G)|$.
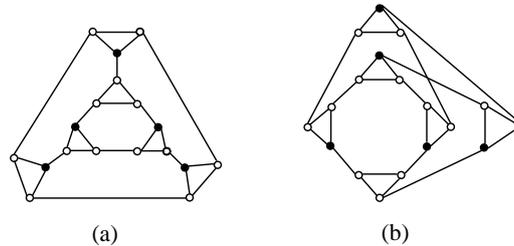


(a)             (b)

**Figure 3.** Two cubic graphs with independent domination number $\frac{5}{18}$ of their orders.

## 3. Proof of Theorem 4 and Theorem 5

First, we will want to prove Theorem 4, which is stated as follows: If $G$ is an SP-1 graph, then $3i(G) \leq n_2(G) + n_3(G)$.

**Proof.** By means of contradiction, let $G$ be a counterexample of an SP-1 graph with a minimum order. Then, $G$ is connected, and $3i(G) > n_2(G) + n_3(G)$. Let $G = (V(G), E(G))$ and $|V(G)| = n$. We proceed further with several claims.

**Claim 1.** $n \geq 7$.

**Proof.** If $n = 3$, $G = K_3$ is not a counterexample. If $n = 4$, $G$ is either $K_4$ or $K_4^-$. In either of cases, we have $3i(G) = 3 \leq n_2(G) + n_3(G)$.

If $n = 5$, let $v$ be the vertex of degree three, $v_1$, $v_2$, let $v_3$ be the three neighbors of $v$ in $G$, and let $w$ be the vertex not adjacent to $v$. Since $G$ is an SP-1 graph, $w$ belongs to a triangle, i.e., at least two vertices in $\{v_1, v_2, v_3\}$ with $w$ will form triangles. Without loss of generality, it could be assumed that $v_1$, $v_2$, and $w$ form a triangle. Then, $G[\{v, v_1, v_2, w\}]$ is diamond unit since $\Delta(G) = 3$, $v_1 v_3$, $v_2 v_3 \notin E(G)$. That is, the triangle incident $v_3$ does not contain $v_1$ or $v_2$. Then, the unique possibility is that $vwv_3$ forms a triangle. However, it contradicts that $vw \notin E(G)$, i.e., $v_3$ does not belong to any triangle, showing a contradiction. Hence, there is no SP-1 graph when $n = 5$.

If $n = 6$, then $G$ is one of the three graphs shown in Figure 4. In each of the cases, $3i(G) = 6 \leq n_2(G) + n_3(G)$. Therefore, $n \geq 7$. $\qquad\square$
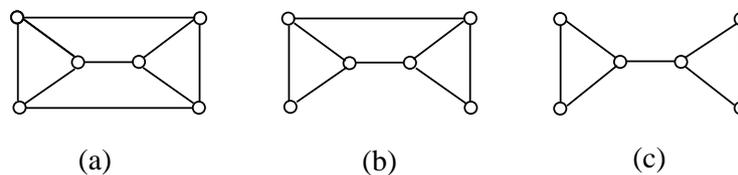


(a)          (b)          (c)

**Figure 4.** The three special subcubic graphs of order 6.

**Claim 2.** Let $X \subseteq V(G)$, $G' = G - X$. If every component of $G'$ is an SP-1 graph with an order of less $n$, then $3i(G') \leq n_2(G') + n_3(G')$.

**Proof.** Let $G_1, G_2, \ldots, G_k$ be the components of $G'$, and let $D_i$ be an $i(G_i)$-set, $i \in [k]$. By linearity, we have

$$3i(G') = 3\sum_{i=1}^{k} i(G_i) \leq \sum_{i=1}^{k}(n_2(G_i) + n_3(G_i)) = n_2(G') + n_3(G').$$

$\qquad\square$

**Claim 3.** $G$ contains no diamond unit.

**Proof.** Suppose that $G$ has a diamond unit $S$, where $V(S) = \{a, b, c, d\}$ and $ab$ are the missing edges. Since $G$ is connected and $n \geq 7$, at least one of $a, b$ has a degree of three.

If $a$ or $b$ has a degree of two, without loss of generality, say $d(b) = 2$. Then, $d(a) = 3$, and $a$ is adjacent to a triangle. Let $G' = G - V(S)$, and let $D'$ be an $i(G')$-set. Note that $G'$ is still an SP-1 graph, and $D' \cup \{c\}$ is an ID-set of $G$. Therefore, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 4 < 3i(G)$, showing a contradiction.

So, $d(a) = d(b) = 3$, and both of $a$ and $b$ are adjacent to triangles. Let $G' = G - V(S)$, and $D'$ be an $i(G')$-set. Then, $D' \cup \{c\}$ is an ID-set of $G$. Note that $G'$ contains at most two components, each of which is SP-1. By Claim 2, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 4 < 3i(G)$, showing a contradiction. $\square$

**Claim 4.** $G$ contains no $C_6^+$-unit.

**Proof.** Suppose that $G$ has a $C_6^+$-unit $F$, where $V(F) = \{o, f, g, h, p, q\}$, and $ofghpqeo$ has two chords $fq$ and $gp$. At least one of $o$ and $h$ has a degree of three since $n \geq 7$.

If $o$ or $f$ has a degree of two, say $f$, then $d(o) = 3$, and $o$ is adjacent to a triangle. Let $G' = G - V(F)$ and let $D'$ be an $i(G')$-set. Noting that $G'$ is SP-1, and $D' \cup \{f, p\}$ is an ID-set of $G$. Then, $3i(G) \leq 3(i(G') + 2) \leq 6 + n_2(G') + n_3(G') = 6 + n_2(G) + n_3(G) - 6 < 3i(G)$, showing a contradiction. Thus, $d(o) = d(h) = 3$. Let $G' = G - V(F)$ and $D'$ be an $i(G')$-set. Then, $G'$ contains at most two components, each of which is SP-1, and $D' \cup \{f, p\}$ is an ID-set of $G$. By Claim 2, $3i(G) \leq 3(i(G') + 2) \leq 6 + n_2(G') + n_3(G') = 6 + n_2(G) + n_3(G) - 6 < 3i(G)$, showing a contradiction. $\square$

**Claim 5.** $G$ is cubic.

**Proof.** Let $T$ be a triangle containing at least one vertex of degree two. Since $n \geq 7$, $T$ has at most two vertices of degree two.

If $T$ has two vertices, say $y$ and $z$, such that $d(y) = d(z) = 2$, then the graph $G' = G - V(T)$. Let $D'$ be an $i(G')$-set. Note that $G'$ is still an SP-1 graph, and $D' \cup \{y\}$ is an ID-set of $G$. Then, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction.

Next, consider the case that $T$ has a unique vertex of degree two. Let $V(T) = \{x, y, z\}$ and $d(z) = 2$, $d(x) = d(y) = 3$. By Claim 3, $x$ and $y$ do not have an incident triangle other than $T$. Thus, each vertex in $G' = G - V(T)$ is still contained in a triangle. This indicates that each component of $G'$ is also SP-1. Let $D'$ be an $i(G')$-set, then $D' \cup \{z_1\}$ is an ID-set of $G$. Therefore, $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. $\square$

According to Claims 3–5 and the definition of SP-1 graph, $G$ is a $\{K_{1,3}, diamond, C_6^+\}$-free cubic. Let $T_1$ be a triangle in $G$, where $V(T_1) = \{x_1, y_1, z_1\}$. Suppose that $x_2$, $x_3$, and $x_4$ are not on $T_1$ and are the neighbors of $x_1$, $y_1$, and $z_1$, respectively. By Claim 3, $x_2$, $x_3$, and $x_4$ are distinct. Also, by Claim 4, any two of $x_2$, $x_3$, and $x_4$ do not contain a triangle. Therefore, $x_1$, $y_1$ and $z_1$ are adjacent to three different triangles, say $T_2$, $T_3$, and $T_4$, respectively. Let $V(T_i) = \{x_i, y_i, z_i\}$, $i = 2, 3, 4$, and $\{x_1 x_2, y_1 x_3, z_1 x_4\} \subseteq E(G)$. Let $G' = G - V(T_1)$, and let $D'$ be an $i(G')$-set. Note that every component of $G'$ is SP-1. If $\{x_2, x_3, x_4\} \subseteq D'$, then $D'$ is also an ID-set of $G$. Thus, $3i(G) \leq 3i(G') \leq n_2(G') + n_3(G') = n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. So, without loss of generality, it could be assumed $x_2 \notin D'$. Then, $D' \cup \{x_1\}$ is an ID-set of $G$, and $3i(G) \leq 3(i(G') + 1) \leq 3 + n_2(G') + n_3(G') = 3 + n_2(G) + n_3(G) - 3 < 3i(G)$, showing a contradiction. $\square$

Now we give the proof of Theorem 5.

**Proof.** Let $n = |V(G)|$, since $G$ is a claw-free cubic graph, then $G$ is an SP-1 graph, and each vertex of $V(G)$ has a degree of 3. Thus, by Theorem 4, $i(G) \leq \frac{1}{3}|V(G)|$.

Next, the extreme graphs are described, i.e., $i(G) = \frac{n}{3}$. Since $G$ is a cubic graph, $n$ is even. When $n = 4$, $G = K_4$, and $i(G) = \frac{n}{4}$, there is a contradiction. When $n = 6$, $G$ is either $C_3 \square K_2$ or $K_{3,3}$, and since $G$ is claw-free, we have $G = C_3 \square K_2$; in this case, $G \in \mathcal{H}$. Thus, $n \geq 8$.

**Claim 1.** $G$ has no diamond unit.

**Proof.** Suppose there is a diamond unit $S$, where $V(S) = \{a, b, c, d\}$ and $ab$ is the missing edge in $S$. Let $x$ be the neighbor of $a$ not in $S$, and let $y$ be the neighbor of $b$ not in $S$. Then, $x \neq y$, and otherwise, $x$ does not belong to a triangle unit.

If $x$ and $y$ are not adjacent, then $x$ and $y$ belong to different triangles. Let $G'$ be obtained from $G - V(S)$ by adding the flat edge $xy$. Thus, $G'$ is an SP-1 cubic graph. Let $D'$ be an $i(G')$-set of $G'$, so $|D'| = i(G') \leq \frac{n-4}{3}$. At most one of $x$ and $y$ is contained in $D'$ since $xy \in E(G)$. If $x \in D'$ and $y \notin D'$, let $D = D \cup \{b\}$. If $y \in D'$ and $x \notin D'$, let $D = D' \cup \{a\}$. If $x, y \notin D'$, let $D = D' \cup \{c\}$. In each of these cases, the set $D$ is an ID-set of $G$, and $|D| = |D'| + 1$, indicating that $i(G) \leq |D'| + 1 \leq \frac{n-1}{3} < \frac{n}{3}$, a contradiction.

If $x$ and $y$ are adjacent in $G$, then they belong to a common triangle unit $T$ in $G$. Let $G' = G - V(S)$, and $G'$ is an SP-1 graph. Let $D'$ be an $i(G')$-set of $G'$, $|D'| = i(G') \leq \frac{n-4}{3}$. Note that $D' \cup \{c\}$ is an ID-set of $G$, so $i(G) \leq |D| = |D| + 1 \leq \frac{n-4}{3} + 1 = \frac{n-1}{3} < \frac{n}{3}$, showing a contradiction. Thus, $G$ has no diamond unit. $\square$

**Claim 2.** Every triangle unit in $G$ belongs to a $C_6^+$-unit.

**Proof.** Suppose that a triangle unit $T_1$ is not in any $C_6^+$-unit, where $V(T_1) = \{x_1, y_1, z_1\}$. Since $G$ is diamond-free and $n \geq 8$, then any two vertices in $V(T_1)$ have no common neighbor except the vertex in $V(T_1)$. Meanwhile, since $T_1$ is not contained in any $C_6^+$-unit, then any two vertices in $V(T_1)$ are not adjacent to a common triangle except for $T_1$. Thus, it is assumed that $x_1$ adjacent to a triangle $T_2$, $y_1$ is adjacent to $T_3$, and $z_1$ is adjacent to $T_4$, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3, 4\}$, and $\{x_1 x_2, y_1 x_3, z_1 x_4\} \subseteq E(G)$. Let $W = \{y_2, z_2, y_3, z_3, y_4, z_4\}$. This proof will be completed by considering the following three cases.

*Case* 1. $e(G[W]) \geq 5$. If $e(G[W]) = 6$, then $G$ is determined, and $i(G) = 3 = \frac{n}{4}$, contradicting that $i(G) = \frac{n}{3}$. Therefore, $e(G[W]) = 5$. Let $e_1$ and $e_2$ be two edges in $G[W]$ and not be $y_2 z_2$, $y_3 z_3$, or $y_4 z_4$. There are two distributions of $e_1$ and $e_2$, as demonstrated in Figure 5. Let $G' = G - \cup_{i=1}^4 V(T_i)$, and let $D'$ be an $i(G')$-set, so every components of $G'$ is SP-1. We have $i(G') \leq \frac{1}{3}(n - 12) = \frac{n}{3} - 4$ by Theorem 4. Note that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of $G$. Thus, $i(G) \leq D' + 3 = \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction.
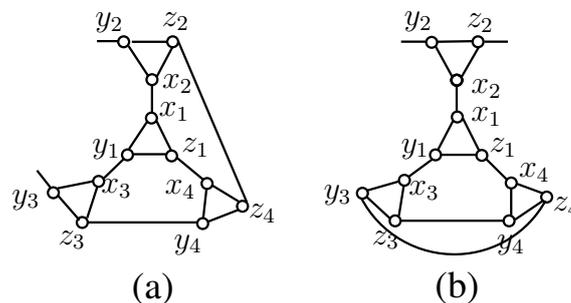


**Figure 5.** The graphs for Case 1 when $e(G[W]) = 5$.

*Case* 2. $e(G[W]) = 4$. By symmetry, it is assumed that $z_3 y_4 \in E(G)$. Let $x$ be the neighbor of $y_2$ not in $W$, let $y$ be the neighbor of $z_2$ not in $W$, let $z$ be the neighbor of $y_3$ not in $W$, and let $r$ be the neighbor of $z_4$ not in $W$. Since $d(x) = 3$, at least one vertex in $\{y, z, r\}$ is not adjacent to $x$, and it is assumed that $xr \notin E(G)$. Let $G'$ be obtained from $G - \cup_{i=1}^4 V(T_i)$ by adding a flat edge $xr$. Thus, every component of $G'$ is SP-1. Let $D'$ be an $i(G')$-set of $G'$. Therefore, $i(G') = |D'| \leq \frac{1}{3}(n - 12) = \frac{n}{3} - 4$. Note that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of $G$, so $i(G) \leq \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction.

*Case* 3. $e(G[W]) = 3$. Let $y_i'$ be the neighbor of $y_i$, and let $z_i'$ be the neighbor of $z_i$, $i \in \{2, 3, 4\}$. Let $G' = G - \cup_{i=1}^4 V(T_i)$. If $G'$ is connected, then $G'$ is an SP-1 graph, and $i(G') \leq \frac{n-12}{3} = \frac{n}{3} - 4$. If $G'$ is disconnected, then it is assumed that $G'$ consists of $k$ components, and $G_1, G_2, \ldots, G_k$, so each component is SP-1. Thus, by Theorem 4, $i(G') = \sum_{i=1}^k i(G_i) \leq \sum_{i=1}^k \frac{1}{3}(n_2(G_i) + n_3(G_i)) = \frac{1}{3}(n_2(G') + n_3(G'))$. Let $D'$ be an $i(G')$-set of $G'$. Then, $D'$ can be extended to an ID-set of $G$ by adding to it the vertices $x_2$, $x_3$, and $x_4$. It implies that $i(G) \leq |D'| + 3 \leq \frac{n}{3} - 4 + 3 = \frac{n}{3} - 1 < \frac{n}{3}$, showing a contradiction. $\square$

Hence, every triangle unit belongs to a $C_6^+$-unit, i.e., the vertex $V(G)$ can be partitioned into sets each of which induces a $C_6^+$-unit in $G$. That is, $G \in \mathcal{H}$. $\square$

## 4. Proof of Theorem 6

In this section, we give the proof of Theorem 6. Now we recall the content of Theorem 6: If $G$ is an SP-2 graph without $(C_3 \square K_2)$-component, then $18i(G) \leq 8n_2(G) + 5n_3(G)$.

**Proof.** Let $G = (V(G), E(G))$ be a counterexample SP-2 graph to Theorem 6 with a minimum order. Apparently, $G$ is connected, and $18i(G) > 8n_2(G) + 5n_3(G)$. If $n = 4$, then $G$ is $K_4$, which is not a counterexample. Similar to the argument of Theorem 4, there is no SP-2 graph when $n = 5$. If $n = 6$, $G$ is one of the three graphs, as shown in Figure 4. In each of the cases, $18i(G) = 36 \leq 8n_2(G) + 5n_3(G)$. Therefore, $n \geq 7$.

Then, the following useful fact is proved.

**Fact 1.** Let $X \subseteq V(G)$, $G' = G - X$, and let $D'$ be an $i(G')$-set. If $G'$ is an SP-2 graph, and there exists a set $D$ such that $D' \cup D$ is an ID-set of $G$, then $18|D| > 8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G'))$.

**Proof.** By means of contradiction, it is assumed that $18|D| \leq 8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G'))$. Since $D \cup D'$ is an ID-set of $G$ and by the minimality of $G$, we have

$$18i(G) \leq 18(|D| + |D'|) \leq 18|D| + 8n_2(G') + 5n_3(G') \leq 8n_2(G) + 5n_3(G).$$

It contradicts that $G$ is a counterexample. □

The following claims are provided to describe some structural properties of $G$.

**Claim 1.** The following properties hold in $G$.

(1) The removal of flat edges of $G$ cannot create an induced $K_{1,3}$, $K_4^-$, or $C_6^+$ subgraph;

(2) Adding flat edges on $G$ to obtain a result graph with a maximum degree of three cannot create an induced $K_{1,3}$ or $K_4^-$ subgraph.

**Proof.** (1) Let $G'$ be obtained from $G$ by removing some flat edges. Note that each vertex in $V(G')$ belongs to a triangle unit, and $\Delta(G) \leq 3$, so $G'$ has no induced $K_{1,3}$. To the contrary, suppose that $G'$ has a $K_4^-$-unit, say $S$, where $V(S) = \{a, b, c, d\}$ and $ab$ is the missing edge in $S$. Since $G$ has no $K_4^+$-unit, then $ab \in E(G)$, and $ab$ is the removing flat edge. Thus, $G$ is determined, and $G = K_4$, contradicting that $n \geq 7$. Similarly, if $G'$ has a $C_6^+$-unit, then $G = C_3 \square P_2$, showing a contradiction.

(2) Let $G'$ be obtained from $G$ by adding some flat edges on $G$, $\Delta(G') = 3$. Since each vertex in $V(G')$ is still in a triangle unit and each edge in a $K_4^-$-unit is a triangle edge, then $G'$ is $\{K_{1,3}, K_4^-\}$-free. □

**Claim 2.** Any two triangle units have no common vertex.

**Proof.** Suppose $T_1$ and $T_2$ are two triangles with common vertices. If they have three common vertices, then $V(T_1) = V(T_2)$. If they have two common vertices, then $G = K_4$, or $G$ has a $K_4^-$-unit, showing a contradiction. If $T_1$ and $T_2$ have only one common vertex, say $x$, then $d(x) \geq 4$, contradicting that $\Delta(G) = 3$. □

**Claim 3.** No triangle in $G$ has two vertices of degree two.

**Proof.** Suppose that there is a triangle unit $T_1$ with two vertices of degree two (see Figure 6a). Let $V(T_1) = \{x_1, y_1, z_1\}$, $d(y_1) = d(z_1) = 2$. Since $n \geq 7$, $d(x_1) = 3$, and $x_1$ is adjacent to a triangle $T_2$. Also, at most one of $V(T_2)$ has a degree of two since $n \geq 7$. Let $X = V(T_1)$, $G' = G - X$, so $G'$ is connected and $\Delta(G) = 3$. Therefore, by Claim 1, $G'$ is an SP-2 graph. Let $D'$ be an $i(G')$-set of $G'$, $D = \{y_1\}$, and then $D' \cup D$ is an ID-set of $G$. However, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 18 = 18|D|$, contradicting to Fact 1. □

**Claim 4.** There are no 3 consecutive triangles with vertex of degree 2 in $G$.

**Proof.** Suppose $T_1$, $T_2$ and $T_3$ are three consecutive triangles with a vertex of degree two (see Figure 6b), where $V(T_i) = \{x_i, y_i, z_i\}$ and $d(x_i) = 2$, $i \in [3]$. If $y_1$ and $z_3$ are adjacent, then the graph $G$ is determined. In this case, $18i(G) = 54 = 8n_2(G) + 5n_3(G)$, contradicting the fact that $G$ is a counterexample to our theorem. Thus, $y_1 z_3 \notin E(G)$. Let $G'$ be the graph obtained from $G - V(T_2)$ by adding the edge $z_1 y_3$. By Claim 1 and $\Delta(G') = 3$, $G'$ is SP-2. Let $D'$ be an $i(G')$-set of $G'$. If $z_1$ and $y_3$ are not contained in $D'$, then $D' \cup \{x_2\}$ is an ID-set of $G$; otherwise, say $z_1 \in D'$, then $D' \cup \{z_2\}$ is an ID-set of $G$. This suggests that $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction. □
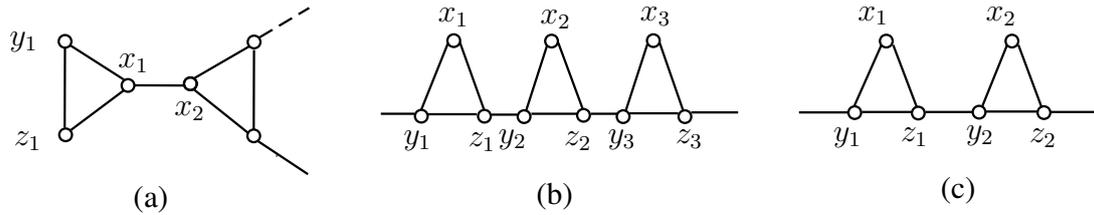
**Figure 6.** The graphs for Claims 3–5.

**Claim 5.** There are no two consecutive triangles with a vertex of degree two in $G$.

**Proof.** Suppose $T_1$ and $T_2$ are two consecutive triangles with a vertex of degree two (see Figure 6c), where $V(T_i) = \{x_i, y_i, z_i\}$ and $d(x_i) = 2$, $i \in [2]$. Since $n \geq 7$, $y_1 z_2 \notin E(G)$.

If $y_1$ and $z_2$ are adjacent to the same triangle (see Figure 7a), say $T_3$, where $V(T_3) = \{x_3, y_3, z_3\}$, $\{y_1 y_3, z_2 z_3\} \subseteq E(G)$, then $V(T_3)$ has no vertex of degree two by Claim 4, and $x_3$ is adjacent to another triangle $T_4$, where $V(T_4) = \{x_4, y_4, z_4\}$, $x_3 x_4 \in E(G)$. Furthermore, $T_4$ has at most one vertex of degree two by Claim 3. In this case, let $X = \cup_{i=1}^4 V(T_i)$, $G' = G - X$, and let $D'$ be an $i(G')$-set of $G'$. Let $D = \{y_1, z_2, x_4\}$, and then $D \cup D'$ is an ID-set of $G$. Thus, if $V(T_4)$ has a vertex of degree two, then $G'$ is a connected SP-2 graph, and $\Delta(G') = 3$, so $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 66 > 18|D|$, contradicting to Fact 1. Thus, each of $V(T_4)$ has a degree of three. If $G'$ is not an SP-2 graph, then $G' = K_3$, and further there is a $C_6^+$-unit in $G$, a contradiction to the assumption that $G$ is an SP-2 graph. Thus, $G'$ is an SP-2 graph, and then $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 18|D|$. This contradicts Fact 1.
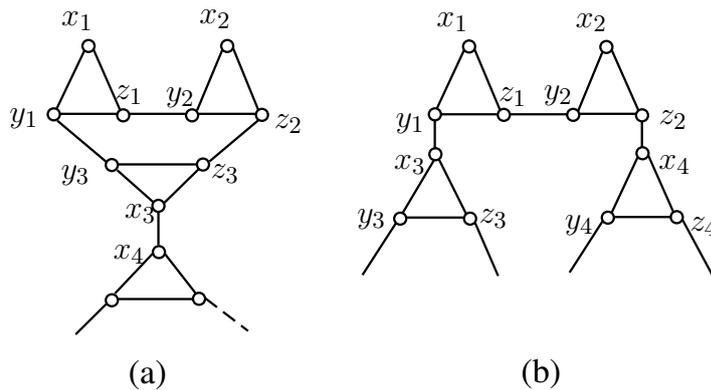


**Figure 7.** The graphs for Claim 5.

If $y_1$ is adjacent to a triangle $T_3$, $z_2$ is adjacent to a triangle $T_4$, $T_3 \neq T_4$ (see Figure 7b), where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3\}$, and $\{y_1 x_3, z_2 x_4\} \subseteq E(G)$. In this case, let $G'$ be obtained from $G - V(T_1)$ by adding the edge $x_3 y_2$. Therefore, $G'$ is an SP-2 graph, and $G \neq C_3 \square P_2$. Let $D'$ be an $i(G')$-set of $G'$. If $x_3$ and $y_2$ are not in $D'$, then $D' \cup \{x_1\}$ is an ID-set of $G$; otherwise, say $x_3 \in D'$, then $D' \cup \{z_1\}$ is an ID-set of $G$. Thus, $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction. $\square$

**Claim 6.** $G$ is cubic.

**Proof.** Assume a triangle $T_1$, where $V(T_1) = \{x_1, y_1, z_1\}$, contains a vertex $x_1$ of degree of two (see Figure 8a). Then, $y_1$ and $z_1$ have no common neighbor except for $x_1$ since $G$ has no diamond unit and $n \geq 7$. Furthermore, they are not adjacent to the same triangle since $G$ has no $C_6^+$-unit. Thus, it is assumed that $y_1$ and $z_1$ are adjacent to $T_2$ and $T_3$, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3\}$, $\{y_1 x_2, z_1 x_3\} \subseteq E(G)$. Each of $V(T_2 \cup T_3)$ has a degree of three by Claim 5.

If both $y_2$ and $z_2$ are not adjacent to $y_3$ or $z_3$, let $G'$ be obtained from $G - V(T_1)$ by adding the edge $x_2 x_3$. Thus, $G'$ is an SP-2 graph, and $G \neq C_3 \square P_2$. Let $D'$ be an $i(G')$-set of $G'$. If $x_2$ and $x_3$ are not in $D'$, then $D' \cup \{x_1\}$ is an ID-set of $G$; otherwise, say $x_2 \in D'$, then $D' \cup \{z_1\}$ is an ID-set of $G$. Thus, $18i(G) \leq 18(i(G') + 1) \leq 18 + 8(n_2(G) - 1) + 5(n_3(G) - 2) < 18i(G)$, showing a contradiction.

Without loss of generality, it is assumed that $z_2 y_3 \in E(G)$ (see Figure 8b), and then $y_2 z_3 \notin E(G)$ since $G$ has no $C_6^+$-unit. $y_2$ and $z_3$ are adjacent to the same triangle, say $T_4$, where $V(T_4) = \{x_4, y_4, z_4\}$, $\{y_2 y_4, z_3 z_4\} \subseteq E(G)$. If $d(x_4) = 2$, then graph $G$ is determined. In this case, $8n_2(G) + 5n_3(G) = 66 > 54 = 18i(G)$, contradicting the fact that $G$ is a counterexample to our theorem. Thus, $d(x_4) = 3$. Let $X = \cup_{i=1}^4 V(T_i)$, $G' = G - X$, and then $G'$ is an SP-2 graph by Claim 2 and Claim 3. Let $D = \{y_1, y_3, y_4\}$ and $D'$ be an $i(G')$-set of $G'$. Hence, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 54 = 18|D|$, showing a contradiction. So, it is assumed that $y_2$ is adjacent to $T_4$, and $z_3$ is adjacent to $T_5$ (see Figure 8c), where $V(T_4) = \{x_4, y_4, z_4\}$, $V(T_5) = \{x_5, y_5, z_5\}$, $y_2 x_4, z_3 x_5 \in E(G)$. Let $X = \cup_{i=1}^4 V(T_i)$, $G' = G - X$, and let $D = \{y_1, y_3, x_4\}$, $D'$ be an $i(G')$-set of $G'$. Thus, if $V(T_4)$ has a vertex of degree two, $G'$ is an SP-2 graph without $C_3 \square K_2$ component, so $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 60 > 54 = 18|D|$, showing a contradiction. Thus, each of $V(T_4)$ has a degree of 3. If $G'$ has no $K_3$ component, then $G'$ is SP-2. In this case, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = 54 = 18|D|$, showing a contradiction. If $G'$ has a $K_3$ component, i.e., $y_4$ and $z_4$ are adjacent to the same triangle $T_6$, then $G$ contains a $C_6^+$-unit, showing a contradiction.  $\square$
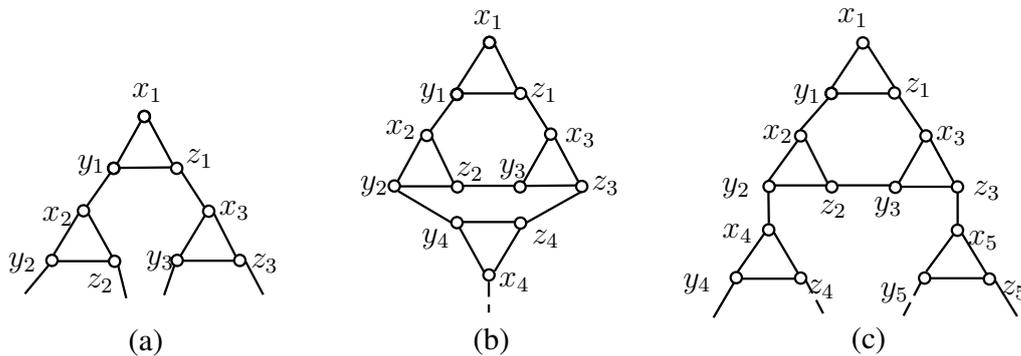


**Figure 8.** The graphs for Claim 6.

**Claim 7.** $G$ contains no $C_6$ subgraph.

**Proof.** Suppose that $G$ contains a $C_6$, $V(C_6) = \{v_1, v_2, \ldots, v_6\}$. The following two cases are considered.

*Case* 1. There exist triangles formed by the inner vertices in $V(C_6)$. Suppose that $T_1$ is a triangle, where $V(T_1) = \{v_1, v_2, v_3\}$. If the triangle where $v_4$ is located is also formed by three vertices inside $V(C_6)$, then $v_2 v_4 \in E(G)$ or $v_4 v_6 \in E(G)$. If $v_2 v_4 \in E(G)$, then $G[\{v_1, v_2, v_3, v_4\}]$ is a $K_4^-$-unit in $G$, showing a contradiction. If $v_4 v_6 \in E(G)$, then $G[V(C_6)]$ is a $C_6^+$-unit, showing a contradiction. Thus, the triangle where $v_4$ is located consists of $v_4$, $v_5$, and a vertex outside $V(C_6)$ by Claim 2. Therefore, $v_6$ does not belong to a triangle, showing a contradiction.

*Case* 2. There is no triangle formed by the inner vertices inside $V(C_6)$. Hence, flat and triangle edges alternate along $C_6$ (see Figure 9a). Suppose $T_1$, $T_2$, and $T_3$ are three triangles containing vertex in $V(C_6)$, where $V(T_i) = \{x_i, y_i, z_i\}$ and $x_i$ is a vertex outside $C_6$, $i \in [3]$. Given that $G$ is $C_6^+$-free, then $x_i x_{i+1} \notin E(G)$, where $i + 1$ is understand modulo 3, $i \in [3]$. If $x_i$ and $x_{i+1}$ have a common neighbor, say $x$, then $x$ is not in a triangle, showing a contradiction.

Next, it is claimed that $x_i$ and $x_{i+1}$ are not adjacent to the same triangle. By means of contradiction, suppose $x_1$ and $x_3$ are adjacent to the same triangle, say $T_4$ (see Figure 9b), where $V(T_4) = \{x_4, y_4, z_4\}$, and $x_1 y_4, x_3 z_4 \in E(G)$. If $x_2 x_4 \in E(G)$, then $G$ is determined, and $18i(G) \leq 54 < 5n_3(G) = 60$, contradicting that $G$ is a counterexample graph. If $x_2 x_4 \notin E(G)$, let $G' = G - \cup_{i=1}^4 V(T_i)$, and $D'$ is an $i(G')$-set of $G'$. By Claim 1, $G'$ is an SP-2 graph. Let $D = \{z_2, y_3, y_4\}$, and then $D \cup D'$ is an ID-set of $G$. Thus, $8(n_2(G) - n_2(G')) + 5(n_3(G) - n_3(G')) = -16 + 70 = 54 = 18|D|$, showing a contradiction.

So, $x_1$, $x_2$, and $x_3$ are adjacent to different triangles, respectively. Assume that $x_i$ is adjacent to $T_{i+3}$, where $V(T_{i+3}) = \{x_{i+3}, y_{i+3}, z_{i+3}\}$ and $x_i x_{i+3} \in E(G)$, $i \in [3]$ (see Figure 9c). If both $y_5$ and $z_5$ are not adjacent to $y_6$ and $z_6$, let $G'$ be obtained from $G - \cup_{i=1}^4 V(T_i)$ by adding the flat edge $x_5 x_6$. Note that $G'$ is SP-2 and $G' \neq C_3 \square P_2$. Every $i(G')$-set of $G'$ can be extended to be an ID-set of $G$ by adding to it the vertices $x_4$, $z_5$ and $y_6$, Therefore, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) - 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. Thus, it is assumed that $z_5 y_6 \in E(G)$, and then $z_6 y_5 \notin E(G)$; otherwise, $G[V(T_5) \cup V(T_6)]$ is a $C_6^+$-unit. By symmetry, $y_4 y_5 \in E(G)$, and $z_4 z_6 \in E(G)$. Therefore, $G$ is determined. In this case, $18i(G) = 90 = 5n_3(G)$, contradicting the fact that $G$ is a counterexample to our theorem.  $\square$
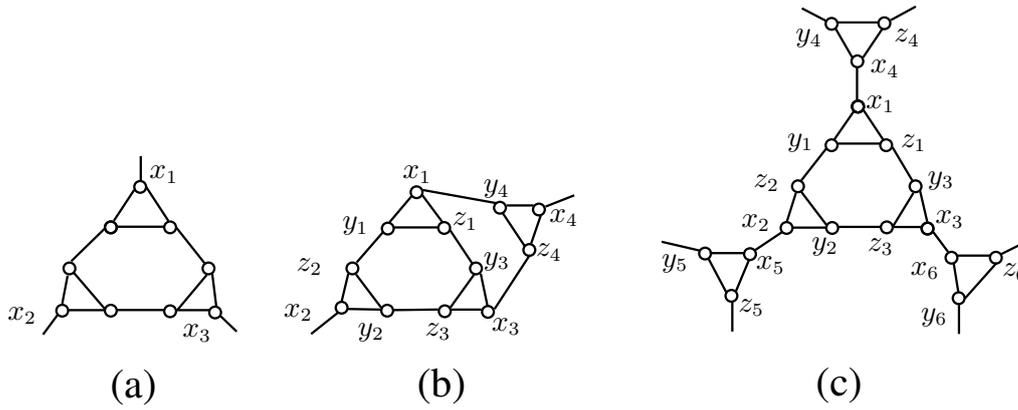
**Figure 9.** The graphs for Claim 7.

**Claim 8.** $G$ contains no $C_8$ subgraph.

**Proof.** Suppose $G$ contains a $C_8$, the following two cases are considered.

*Case* 1. There exist triangles formed by the inner vertices in $V(C_8)$. Since $G$ has no $K_4$-unit, there are at most two disjoint triangles formed by $V(C_8)$.

*Case* 1.1. If there are two triangles formed by the inner vertices in $V(C_8)$, then there is a $C_6$ in $G[V(C_8)]$, showing a contradiction.

*Case* 1.2. There is a unique triangle $T$ formed by the inner vertices in $V(C_8)$. For the renaming vertices except for $T$, we have $\{v_1, v_2, \ldots, v_5\} = V(C_8) \setminus V(T)$. For any $v_i \in V(C_8) \setminus V(T)$, the triangle where $v_i$ is located cannot only contain $v_i$ in $V(C_8)$; otherwise, $d(v_i) \geq 4$. By Claim 2, any two triangles have no common vertex, but it is impossible to have a partition in $V(C_8) \setminus V(T)$ such that each part has two vertices, showing a contradiction.

*Case* 2. There is no triangle formed by the inner vertices inside $V(C_8)$. That is, $V(C_8)$ is partitioned into 4 binary sets (see Figure 10a). In each binary set, these two vertices are adjacent, and otherwise, there is at least one $4^+$-vertex, showing a contradiction. Any two vertices in $\{x_1, x_2, x_3, x_4\}$ are not adjacent since $G$ is $C_6$-free.
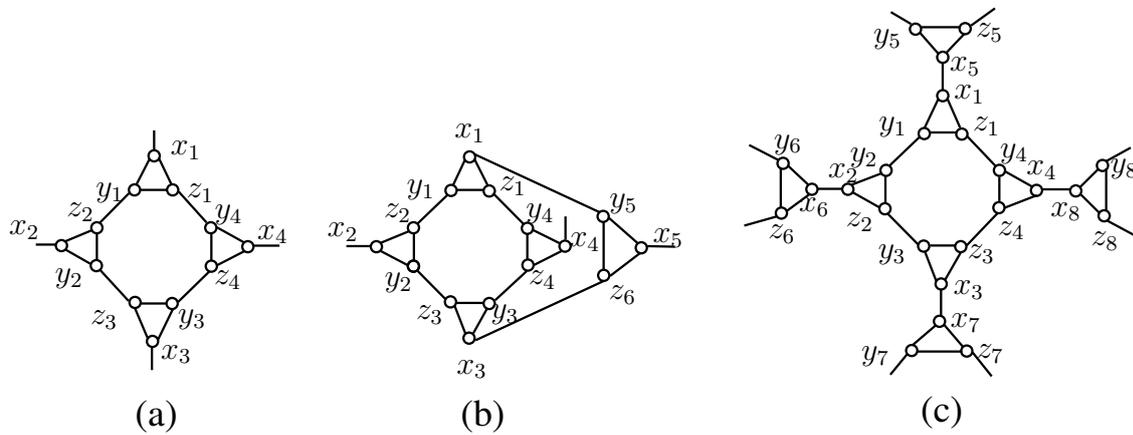


**Figure 10.** The graphs for Claim 8.

**Claim 8.1.** Any two vertices in $\{x_1, x_2, x_3, x_4\}$ are not adjacent to the same triangle.

**Proof.** First, $x_i$ and $x_{i+1}$ are not adjacent to the same triangle because $G$ is $C_6$-free. By the symmetry, it is assumed that $x_1$ and $x_3$ are adjacent to the same triangle $T_5$ (see Figure 10b), where $V(T_5) = \{x_5, y_5, z_5\}$, $x_1 y_5, x_3 z_5 \in E(G)$. Note that $x_2 x_5, x_4 x_5 \notin E(G)$, since $G$ has no $C_6$ subgraph.

*Case* A. $x_5, x_2, x_4$ are adjacent to the same triangle. Then, $G$ is determined, and $i(G) = 5$, so $18i(G) = 90 = 5n$, showing a contradiction.

*Case* B. $x_5$ and $x_2$ are adjacent to the same triangle, and $x_4$ is adjacent to another triangle. Assume that $x_2$ and $x_5$ are adjacent to the same triangle $T_6$, where $V(T_6) = \{x_6, y_6, z_6\}$, $\{x_2 y_6, x_5 z_6\} \subseteq E(G)$. Meanwhile, $x_4$ is adjacent to a triangle $T_7$, $T_7 \neq T_6$, where $V(T_7) = \{x_7, y_7, z_7\}$, $x_4 x_7 \in E(G)$. Let $G'$ be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$ by adding the flat edges $y_3 y_6, x_3 x_7$, and let $D'$ be an $i(G')$-set of $G'$. Since

both $T_6$ and $T_7$ are not adjacent to $T_3$, $G'$ has no $C_6^+$-unit. Hence, $G'$ is an SP-2 graph, and $D' \cup \{z_2, y_4, y_5\}$ is an ID-set of $G$. Thus, $18i(G) \leq 18(i(G')+3) \leq 54+8(n_2(G)+2)+5(n_3(G)-14) = 8n_2(G)+5n_3(G) < 18i(G)$, showing a contradiction.

*Case* C. $x_5$ is not adjacent to the same triangle with $x_2$ or $x_4$. Assume that $x_2$ is adjacent to a triangle $T_6$, and $x_5$ is adjacent to a triangle $T_7$, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{6, 7\}$. Let $G'$ be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$, and let $D'$ be an $i(G)$-set of $G'$. Since both $T_6$ and $T_7$ are not adjacent to $T_3$, $G'$ has no $C_6^+$-unit. Hence, $G'$ is an SP-2 graph, and $D' \cup \{z_2, y_4, x_5\}$ is an ID-set of $G$. Therefore, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) + 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. $\square$

Thus, it is assumed that $x_i$ is adjacent to triangle $T_{i+4}$, where $V(T_{i+4}) = \{x_{i+4}, y_{i+4}, z_{i+4}\}$, $i \in [4]$, $\{x_1x_5, x_2x_6, x_3x_7, x_4x_8\} \subseteq E(G)$ (see Figure 10c). Let $G'$ be obtained from $G - V(T_1) \cup V(T_2) \cup V(T_4) \cup V(T_5)$ by adding the flat edges $y_3x_6, z_3x_8$, and $D'$ is an $i(G')$-set of $G'$. Since both $T_6$ and $T_8$ are not adjacent to $T_3$, $G'$ has no $C_6^+$-unit. Thus, $G'$ is an SP-2 graph.

Hence, $18i(G) \leq 18(i(G') + 3) \leq 54 + 8(n_2(G) + 2) + 5(n_3(G) - 14) = 8n_2(G) + 5n_3(G) < 18i(G)$, showing a contradiction. $\square$

Therefore, $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph, and $G$ has no $C_6$ or $C_8$ as subgraph. Fix a triangle $T_1 \subseteq V(G)$, where $V(T_1) = \{x_1, y_1, z_1\}$ (see Figure 11). $x_1$, $y_1$, and $z_1$ have no common neighbors because $G$ is $K_4^-$-free. Any two vertices in $V(T_1)$ can not be adjacent to the same triangle since $G$ is $C_6^+$-free and $G \neq C_3 \square P_2$. Hence, it is assumed that $x_1$, $y_1$, and $z_1$ are adjacent to $T_2$, $T_3$, and $T_4$, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{2, 3, 4\}$, and $\{x_1x_2, y_1x_3, z_1x_4\} \subseteq E(G)$. For any $i, j \in \{2, 3, 4\}$, $i \neq j$, both $y_i$ and $z_i$ are not adjacent to $y_j$ or $z_j$ because $G$ has no $C_6$ subgraph. $y_i$ and $z_j$ can not be adjacent to the same triangle because $G$ is $C_8$-free. Thus, $y_2$, $z_2$, $y_3$, $z_3$, $y_4$, and $z_4$ are adjacent to different triangles, respectively. Suppose that $y_2$, $z_2$, $y_3$, $z_3$, $y_4$, and $z_4$ are adjacent to $T_5$ $T_6$, $T_7$, $T_8$, $T_9$, and $T_{10}$, respectively, where $V(T_i) = \{x_i, y_i, z_i\}$, $i \in \{5, 6, 7, 8, 9, 10\}$, $\{y_2x_5, z_2x_6, y_3x_7, z_3x_8, y_4x_9, z_4x_{10}\} \subseteq E(G)$. Let $G'$ be obtained from $G - \cup_{i=1}^4 V(T_i)$, and $D'$ be an $i(G')$-set. By Claim 1 and Claim 7, $G'$ is SP-2. Noting that $D' \cup \{x_2, x_3, x_4\}$ is an ID-set of $G$, so $18i(G) \leq 18(i(G') + 3) \leq 54 + 5(n_3(G) - 12) = 8n_2(G) + 5n_3(G) - 6 < 18i(G)$, showing a contradiction.
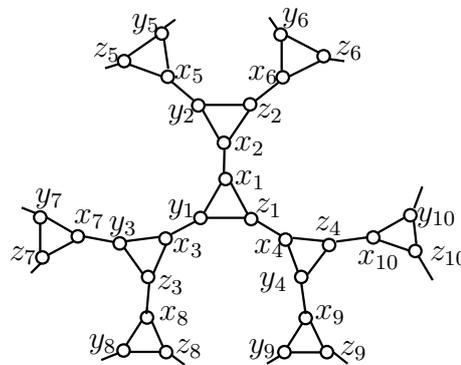


**Figure 11.** Illustrations for $G$.

This completes the proof. $\square$

**Corollary 2.** *If $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \square K_2)$-component, then $i(G) \leq \frac{5}{18}|V(G)|$.*

**Proof.** If $G$ is a $\{K_{1,3}, K_4^-, C_6^+\}$-free cubic graph with no $(C_3 \square K_2)$-component, then $G$ is an SP-2 graph, and each vertex in $V(G)$ is of degree three. Thus, by Theorem 2.3, $i(G) \leq \frac{5}{18}|V(G)|$. $\square$

## Author Contributions

L.L.: conceptualization, methodology, writing; J.Y.: conceptualization, methodology, writing. All authors have read and agreed to the published version of the manuscript.

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The authors declare no conflict of interest.

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