Highlights

Linear State Estimation for Multi-rate NCSs With Multi-channel Observation Delays and Unknown Markov Packet Losses

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Abstract: This paper is concerned with the linear minimum mean square error estimation (LMMSE) for the multi-rate sampling systems with multi-channel observation delays and unknown Markovian packet losses. The original system is firstly transformed into a single-rate jumping parameter system with multi-channel and delay-free observations by employing the lifting technique and introducing a set of reorganized observations and Markov chains. Then, the single-rate system is converted into a general linear system without delays by defining a new group of extended states. Based on the innovation analysis method, a liner minimum mean square error estimator is developed, and the estimator gain is obtained in terms of generalized Riccati difference equations based on a set of coupled Lyapunov equations. Therefore, the original state estimation problem is solved via the jumping parameter property. Finally, the convergence of the Riccati equation is analyzed and a stationary filter is obtained. The novelty of this paper lies in the introduction of the reorganized observations and multi-state Markov chains.

Keywords: linear estimation; markov packet losses; observation delay; reorganization observation methods

1. Introduction

With the development and progress of the control science, the networked control system (NCS) has become a popular research topic due to its wide applications in various practical areas, such as fault warning [1], target tracking [2], power grids [3, 4], transportation [5], and energy storage [6]. However, since the NCS transmits information through the communication network, the limited bandwidth inevitably leads to unreliability of the transmission channel, such as transmission delays and data losses [7−11]. In recent years, state estimation for the NCSs has attracted much attention with fruitful results reported in the literature such as the H_∞ filter [12, 13], Kalman filter [14, 15], set membership filter [16, 17], and moving horizon estimator [18, 19].

As it is well known, in the practical industrial systems, the production process is always influenced by nonlinear factors. Nevertheless, in the modeling process of real industrial systems, the system model should be designed as simple as possible to maintain the essential system characteristics and facilitate system analysis and synthesis. Therefore, the linear system model is employed to characterize the basic features of practical systems such as companding signal processors [20], mixing tank systems [21] and direct current motors [22], where the influence of nonlinear factors on the system is not significant. In this paper, we focus on the optimal filtering for linear NCSs with delays and packet losses.

As for state estimation of discrete systems with time delays, some classical techniques have been developed such as the state augmentation method [23], polynomial method [24], reorganized innovation analysis method [25] and linear matrix inequality (LMI) algorithm [26, 27]. For state estimation of systems with packet losses, the packet loss is usually characterized by the Bernoulli process [28] or the Markov process [29], and the estimator design can borrow ideas from the estimator design for Markov jump linear systems (MJLSs) and multiplicative noise systems

[30]. Actually, the NCSs with random transmission delays can also be transformed into systems with packet losses and transmission delays [31−37].

On the other hand, the multi-rate sampling system, which has shown great importance in improving estimation performance and energy conservation, has been applied to replace the traditional NCSs in a wide variety of fields, such as signal storage [38], transmission [39] and processing [40]. Up to now, there has been numerous results for the single-rate sampling systems, but such results cannot be extended to deal with the multi-rate sampling systems directly, which motivates the investigation of state estimation for the multi-rate sampled systems [41−44]. For the multi-rate uniform sampling system, the lifting technique and state augmentation method have been used in [45] to transform the multi-rate system into an equivalent single-rate system, and then an LMMSE has been designed for the augmented state. In [46], a novel "sampled output augmentation" method has been developed to deal with delayed and infrequent primary measurements by designing a multi-rate Kalman filter. The equivalence has been verified between the proposed method and the traditional fixed-lag smoothing method. For the random multi-rate sampling system [47], a new state model has been developed to describe the dynamics of the measurement sampling points during the state update period. By applying the innovation analysis methods, a non-augmented state estimator has been designed which is dependent on the data loss rate.

Unlike the filter design of single-rate sampling systems, the state estimation of multi-rate sampling systems becomes more complicated accounting for the different sampling periods of sensor components. In summary, so far, the state estimation has not been fully studied for NCSs with multi-channel observation delays and multiple unknown Markov packet losses, especially for the unreliable multi-rate sampling system, which is worthy of further studies.

Motivated by the above analysis, the purpose of this paper is to design a linear state estimator for multi-rate sampling systems with unknown multi-channel Markov packet losses and observation delays. A new set of two-state Markov chains is introduced to characterize the multi-channel packet losses. The original multi-rate sampling system is firstly transformed into a single-rate sampling system with packet losses and observation delays by the lifting technique, and further converted into a Markov jumping parameter system with multi-channel and delay-free observations by using the reorganization observation technique and introducing a new set of multi-mode Markov chains. Since the original packet loss is unknown, the realization of the Markov chain in the converted models is unknown. A new set of states is defined which contains both the original system state and the indicator functions of the new Markov chains. Based on such states, an LMMSE is derived on the basis of the innovation analysis method, where the estimator gain is given in light of a set of generalized coupled difference Riccati equations (GCDREs) with a set of coupled Lyapunov equations. So the original state estimation can be obtained through the characteristics of jump parameters. It can be proven that under appropriate assumptions, the GCDREs converge to a set of generalized coupled algebraic Riccati equations (GCAREs), resulting in a stationary filter. The stationary filter gains can be performed off-line. Finally, the effectiveness of this algorithm is verified through a simulation example. The main novelties of the current research work can be summarized as the following aspects.

1) The state estimation problem for multi-rate systems with multi-channel observation delays and unknown Markov packet losses is considered.

2) The lifting technique and the reorganization observation technique are used to transform the original system into a single-rate and delay-free one.

3) A new set of Markov chains is introduced to characterize the multi-channel packet losses.

4) The proposed algorithm is recursive and suitable for offline calculation.

The remainder of this paper is organized as follows. In Section 2, the problem statement and some preliminaries are given. In Section 3, the LMMSE is developed based on the innovation analysis method. In Section 4, the convergence analysis is given and a stationary filter is obtained. Finally, Section 5 provides a numerical example, and Section 6 draws some conclusions.

Notations: Throughout this paper, R^n denotes the n -dimensional Euclidean space, $R^{m \times n}$ represents the norm bounded linear space of all $m \times n$ matrices. Moreover, diag $\{\cdots\}$ means a block-diagonal matrix. For $L \in R^{n \times n}$, L^T stands for the transpose of L. $E(.)$ indicates the mathematical expectation operator, $1_{[.]}$ stands for the Dirac measure, and $\Pi(.)$ means the occurrence probability of an event.

2. Problem Formulation and Preliminaries

2.1. Problem Formulation

Consider the following multi-rate sampling data system with multi-channel Markov packet dropouts and observation delays:

$$x(k+1) = A_o x(k) + B_o w_o(k), \tag{1}$$

$$y_l(kh) = \gamma_l(kh) H_l x((k-l)h) + v_l((k-l)h),$$
$$l = 0, 1, \cdots, r, \tag{2}$$

where $x(k) \in R^n$ is the state sequence, $\gamma_l(kh) \in R^m, l = 0, 1, \cdots, r$ is the measurement dropout sequence, $w_o(k) \in R^p$ is the process noise, and $v_l(kh) \in R^{q_l}, l = 0, 1, \cdots, r$, is the measurement noise in the lth channel. A_o, B_o and $H_l, l = 0, 1, \cdots, r$, are matrices of appropriate dimensions. The random variables $\gamma_l(kh) \in \{0, 1\}$ ($l = 0, 1, \cdots, r$) are employed to describe the observation dropout,, and each $\gamma_l(kh)$ is modeled as a time-homogeneous two-state Markov chain with the following transition probability matrix (TPM):

$$\Pi_l = \begin{bmatrix} 1 - q_l & q_l \\ p_l & 1 - p_l \end{bmatrix}, \quad l = 0, 1, \cdots, r, \tag{3}$$

where $q_l = \Pr\{\gamma_l(kh+h) = 1 \mid \gamma_l(kh) = 0\}$ is the recovery rate and $p_l = \Pr\{\gamma_l(kh+h) = 0 \mid \gamma_l(kh) = 1\}$ is the failure rate. Denote

$$\pi_{l,i}(kh) = \Pr\{\gamma_l(kh) = i - 1\}, i \in \{1, 2\}. \tag{4}$$

Assume that $0 < p_l, q_l < 1$. Then, there exists the limit probability distribution $\pi_l = \begin{bmatrix} \pi_{l,1} & \pi_{l,2} \end{bmatrix}$ with

$$\pi_{l,1} = \frac{p_l}{p_l + q_l}, \quad \pi_{l,2} = \frac{q_l}{p_l + q_l}.$$

Assumption 1: The initial state x_0, $w_o(k)$ and $v_l(kh)$, $l = 0, 1, \cdots, r$, are null mean white noises with covariance matrices P_0, Q_o and R_l ($l = 0, 1, \cdots, r$), respectively. x_0, $w_o(k)$ and $v_l(kh)$ ($l = 0, 1, \cdots, r$) are mutually independent.

Assumption 2: $\{\gamma_l(kh)\}_{l=0}^r$ are mutually independent.

The estimation problem considered in this paper can be formulated as follows.

Problem: Given system (1)-(2), we aim to derive an LMMSE estimator $\hat{x}(k \mid k)$ such that

$$E\{\|x(k) - \hat{x}(k \mid k)\| \mid y(0), \cdots, y(kh)\} \tag{5}$$

is minimized, while the estimator gains are deterministic.

2.2. System Transformation

In this part, we first transform the multi-rate sampling data system into a single-rate system with multi-channel observation delays and packet losses by the lifting technique.

By iterating (1) after h steps, the state update at the measurement sampling instant $k + 1$ can be established as follows:

$$x((k+1)h) = A x(kh) + w(kh), \tag{6}$$

where

$$A = A_o^h, \tag{7}$$

and the new system noise is defined as

$$w(kh) = \sum_{m=0}^{h-1} A_o^m B_o w_o((k+1)h - m - 1). \tag{8}$$

The statistical properties of the system noise can be computed by

$$Q_w = \sum_{m=0}^{h-1} A_o^m B_o Q_o B_o^T \left(A_o^m\right)^T.$$

Next, we employ the reorganization observation method to transform the multi-channel observation delay system into a delay-free one. In view of the definition of $y(kh)$, we rearrange the received multi-channel observations to eliminate the hindrance of the observation delay. A new sequence without delays is constructed as

$$\bar{y}_r(sh) = \begin{bmatrix} y_0(sh) \\ y_1((s+1)h) \\ \vdots \\ y_r((s+r)h) \end{bmatrix}, 0 \leqslant s < k-r, \tag{9}$$

$$\bar{y}_{k-s}(sh) = \begin{bmatrix} y_0(sh) \\ y_1((s+1)h) \\ \vdots \\ y_{k-s}(kh) \end{bmatrix}, k-r \leqslant s \leqslant k. \tag{10}$$

Then, $\bar{y}_r(sh)$ and $\bar{y}_{k-s}(sh)$ are the delay-free observations which satisfy

$$\bar{y}_r(sh) = \Gamma_r(sh)\bar{H}_r x(sh) + \bar{v}_r(sh),$$
$$0 \leqslant s < k-r, \tag{11}$$

$$\bar{y}_{k-s}(sh) = \Gamma_{k-s}(sh)\bar{H}_{k-s}x(sh) + \bar{v}_{k-s}(sh),$$
$$k-r \leqslant s \leqslant k, \tag{12}$$

where

$$\Gamma_r(sh) = \begin{bmatrix} \gamma_0(sh)I & 0 & & 0 & 0 \\ 0 & \gamma_1((s+1)h)I & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \gamma_r((s+r)h)I \end{bmatrix},$$

$$\Gamma_{k-s}(sh) = \begin{bmatrix} \gamma_0(sh)I & 0 & & 0 & 0 \\ 0 & \gamma_1((s+1)h)I & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \gamma_{k-s}(kh)I \end{bmatrix},$$

$$\bar{H}_r = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_r \end{bmatrix}, \quad \bar{H}_{k-s} = \begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{k-s} \end{bmatrix},$$

$$\bar{v}_r(sh) = \begin{bmatrix} v_0(sh) \\ v_1(sh) \\ \vdots \\ v_r(sh) \end{bmatrix}, \quad \bar{v}_{k-s}(sh) = \begin{bmatrix} v_0(sh) \\ v_1(sh) \\ \vdots \\ v_{k-s}(sh) \end{bmatrix},$$

with $\bar{v}_r(sh)$ and $\bar{v}_{k-s}(sh)$ being white noises with zero means and following covariances:

$$\bar{R}_r = \text{diag}\{R_0, \cdots, R_r\},$$
$$\bar{R}_{k-s} = \text{diag}\{R_0, \cdots, R_{k-s}\}.$$

Moreover, we introduce a set of new multi-mode Markov chains to convert the delay-free system (9) and (10) into a Markov jumping parameter system. This method is named as the reorganization Markov chain method.

For $0 \leqslant s < k-r$,

$$\bar{y}_r(sh) = \mathcal{H}_{r,\theta_r(sh)}x(sh) + \bar{v}_r(sh), \tag{13}$$

$$\mathcal{H}_{r,\theta_r(sh)} = \Gamma_r(sh)\bar{H}_r, \tag{14}$$

where the jumping variable is specified by the following equations:

$$\theta_r(sh) = 1 + \sum_{i=1}^{r+1} 2^{i-1}\gamma_{i-1}(sh), \tag{15}$$

$$\theta_r(sh) \in \aleph_r = \left\{ 1, 2, \cdots, 2^{r+1} \right\},$$

with $N_r = 2^{r+1}$.

The TPM for the Markov process $\{\theta_r(sh)\}$ is denoted as $\bar{\Pi}_r = [p_{ij}^{(r)}]$ which is given by

$$\bar{\Pi}_r = \Pi_r \otimes \Pi_{r-1} \otimes \cdots \otimes \Pi_0. \tag{16}$$

Moreover, denote the distribution of $\theta_r(sh)$ as

$$\mu_{r,i}(sh) = \Pr\{\theta_r(sh) = i\}, i \in \aleph_r, \tag{17}$$

$$\mu_r(sh) = \left[\mu_{r,1}(sh), \cdots, \mu_{r,N_r}(sh) \right]^T. \tag{18}$$

$\theta_r(sh)$ has the following unique stationary distribution:

$$\mu_r = \left[u_{r,1}, \cdots, \mu_{r,N_r} \right]^T,$$

which satisfies $\lim_{k \to \infty} \mu_{r,i}(kh) = \mu_{r,i}$ for $i \in \aleph_r$, and

$$\mu_r = \pi_r \otimes \pi_{r-1} \otimes \cdots \otimes \pi_0. \tag{19}$$

For $k - r \leqslant s \leqslant k$,

$$\bar{y}_{k-s}(sh) = \mathcal{H}_{k-s,\theta_{k-s}(sh)} x(sh) + \bar{v}_{k-s}(sh), \tag{20}$$

$$\mathcal{H}_{k-s,\theta_{k-s}(sh)} = \Gamma_{k-s}(sh)\bar{H}_{k-s} \tag{21}$$

where the jumping variables are specified by the following equations:

$$\theta_{k-s}(sh) = 1 + \sum_{i=1}^{k-s+1} 2^{i-1} \gamma_{k-s}(sh), \tag{22}$$

$$\theta_{k-s}(sh) \in \aleph_{k-s} = \left\{ 1, 2, \cdots, 2^{k-s+1} \right\},$$

with $N_{k-s} = 2^{k-s+1}$.

The TPM for the Markov process $\{\theta_{k-s}(s)\}$ is denoted as $\bar{\Pi}_{k-s} = [p_{ij}^{(k-s)}]$ which satisfies

$$\bar{\Pi}_{k-s} = \Pi_{k-s} \otimes \Pi_{k-s-1} \otimes \cdots \otimes \Pi_0. \tag{23}$$

In addition, define the distribution of $\theta_{k-s}(sh)$ as

$$\mu_{k-s,i}(sh) = \Pr\{\theta_{k-s}(sh) = i\}, i \in \aleph_{k-s}, \tag{24}$$

$$\mu_{k-s}(sh) = \left[u_{k-s,1}(sh), \cdots, \mu_{k-s,N_{k-s}}(sh) \right]^T. \tag{25}$$

$\theta_{k-s}(sh)$ has the following unique stationary distribution:

$$\mu_{k-s} = \left[\mu_{k-s,1}, \cdots, \mu_{k-s,N_r} \right]^T,$$

which satisfies $\lim_{k \to \infty} \mu_{k-s,i}(k) = \mu_{k-s,i}$ for $i \in \aleph_{k-s}$, and

$$\mu_{k-s} = \pi_{k-s} \otimes \pi_{k-s-1} \otimes \cdots \otimes \pi_0. \tag{26}$$

3. Design of the LMMSE Filter

Note that the packet loss in each channel is unknown, so the jumping parameters are unknown at the present time. In this paper, we will estimate the state and Markov chains synchronously by introducing the following stochastic variables:

$$z_{r,j}(sh) \triangleq x(sh)1_{\{\theta_r(sh)=j\}},$$
$$0 \leqslant s < k - r, j = 1, \cdots, N_r, \tag{27}$$

$$z_{k-s,j}(sh) \triangleq x(sh)1_{\{\theta_{k-s}(sh)=j\}},$$
$$k-r \leqslant s \leqslant k, j=1,\cdots,N_{k-s}, \tag{28}$$

$$z_r(sh) = \left[\begin{array}{ccc} z_{r,1}^T(sh) & \cdots & z_{r,N_r}^T(sh) \end{array} \right]^T, \tag{29}$$

$$z_{k-s}(sh) = \left[\begin{array}{ccc} z_{k-s,1}^T(sh) & \cdots & z_{k-s,N_{k-s}}^T(sh) \end{array} \right]^T, \tag{30}$$

where $1_{[.]}$ represents the indicator function.

In view of (27) and (28), the new state equation can be written as the following compact form.

(i) For $0 \leqslant s < k-r$,

$$
\begin{aligned}
z_{r,j}(sh+h) &= \sum_{i=1}^{N_r} Az_{r,i}(sh)1_{\{\theta_r(s+1)=j\}} + \sum_{i=1}^{N_r} w(sh)1_{\{\theta_r(s)=i\}}1_{\{\theta_r(s+1)=j\}} \\
&= \sum_{i=1}^{N_r} Az_{r,i}(sh)p_{ij}^r + \sum_{i=1}^{N_r} Az_{r,i}(sh)\left(1_{\{\theta_r(s+1)=j\}} - p_{ij}^r\right) \\
&\quad + \sum_{i=1}^{N_r} w(sh)1_{\{\theta_r(s)=i\}}1_{\{\theta_r(s+1)=j\}}.
\end{aligned} \tag{31}
$$

Then, we have

$$z_r(sh+h) = \bar{A}_r z_r(sh) + \mathcal{B}_{r,1}(sh) + \mathcal{B}_{r,2}(sh), \tag{32}$$

where

$$\bar{A}_r = \bar{\Pi}_r^T \otimes A,$$

$$\mathcal{B}_{r,1}(sh) = \begin{bmatrix} \sum_{i=1}^{N_r} Az_{r,i}(sh)\beta_{i1}^r(sh) \\ \vdots \\ \sum_{i=1}^{N_r} Az_{r,i}(sh)\beta_{iN_r}^r(sh) \end{bmatrix},$$

$$\mathcal{B}_{r,2}(sh) = \begin{bmatrix} \sum_{i=1}^{N_r} w(sh)1_{\{\theta_r(s)=i\}}1_{\{\theta_r(s+1)=1\}} \\ \vdots \\ \sum_{i=1}^{N_r} w(sh)1_{\{\theta_r(s)=i\}}1_{\{\theta_r(s+1)=N_r\}} \end{bmatrix},$$

$$\beta_{ij}^r(sh) = 1_{\{\theta_r(sh+h)=j\}} - p_{ij}^r.$$

(ii) For $k-r \leqslant s \leqslant k$,

$$
\begin{aligned}
z_{k-s,j}(sh+h) &= \sum_{i=1}^{N_{k-s}} Az_{r,i}(sh)p_{ij}^{k-s} + \sum_{i=1}^{N_{k-s}} Az_{k-s,i}(sh)\left(1_{\{\theta_{k-s}(s+1)=j\}} - p_{ij}^{k-s}\right) \\
&\quad + \sum_{i=1}^{N_{k-s}} w(sh)1_{\{\theta_{k-s}(s)=i\}}1_{\{\theta_{k-s}(s+1)=j\}}.
\end{aligned} \tag{33}
$$

Then, we have

$$z_{k-s}(sh+h) = \bar{A}_{k-s}z_{k-s}(sh) + \mathcal{B}_{k-s,1}(sh) + \mathcal{B}_{k-s,2}(sh), \tag{34}$$

where

$$\bar{A}_{k-s} = \bar{\Pi}_{k-s}^T \otimes A,$$

$$\mathcal{B}_{k-s,1}(sh) = \begin{bmatrix} \sum_{i=1}^{N_{k-s}} Az_{k-s,i}(sh)\beta_{i1}^{k-s}(sh) \\ \vdots \\ \sum_{i=1}^{N_{k-s}} Az_{k-s,i}(sh)\beta_{iN_{k-s}}^{k-s}(sh) \end{bmatrix},$$

$$\mathcal{B}_{k-s,2}(sh) = \begin{bmatrix} \sum_{i=1}^{N_{k-s}} w(sh)1_{\{\theta_{k-s}(s)=i\}}1_{\{\theta_{k-s}(s+1)=1\}} \\ \vdots \\ \sum_{i=1}^{N_{k-s}} w(sh)1_{\{\theta_{k-s}(s)=i\}}1_{\{\theta_{k-s}(s+1)=N_{k-s}\}} \end{bmatrix},$$

$$\beta_{ij}^{k-s}(sh) = 1_{\{\theta_{k-s}(sh+h)=j\}} - p_{ij}^{k-s}.$$

Therefore, the newly constructed state can be estimated directly.

The observations are further transformed into the following forms:

$$\bar{y}_r(sh) = \bar{G}_r z_r(sh) + \bar{v}_r(sh), \quad 0 \leqslant s < k-r, \tag{35}$$

$$\bar{y}_{k-s}(sh) = \bar{G}_{k-s} z_{k-s}(sh) + \bar{v}_{k-s}(sh), \quad k-r \leqslant s \leqslant k, \tag{36}$$

with

$$\bar{G}_r \triangleq \begin{bmatrix} \mathcal{H}_{r,1} & \cdots & \mathcal{H}_{r,Nr} \end{bmatrix},$$

$$\bar{G}_{k-s} \triangleq \begin{bmatrix} \mathcal{H}_{k-s,1} & \cdots & \mathcal{H}_{k-s,N_{k-s}} \end{bmatrix}.$$

Thus, the LMMSE estimation problem for system (1)–(2) is transformed into the one for (1), (29) and (30). Before starting the design, some new definitions and notations are introduced.

Definition 3.1: Given the time kh for $0 \leqslant s \leqslant k-r$, the LMMSE predictor $\hat{z}_r(sh \mid sh-h)$ is defined as the projection of $z_r(sh)$ onto the following linear space:

$$L\left\{ \bar{y}_r(sh)|_{0 \leqslant s < k-r} \right\}. \tag{37}$$

For $k-r < s \leqslant k$, the LMMSE predictor $\hat{z}_{k-s}(sh \mid sh-h)$ is defined as the projection of $z_{k-s}(sh)$ onto the following linear space:

$$L\left\{ \bar{y}_{k-s}(sh)|_{k-r < s \leqslant k} \right\}. \tag{38}$$

Definition 3.2: Define the one-step prediction $\hat{\bar{y}}_r(sh \mid sh-h)$ as the projection of $\bar{y}_r(sh)$ onto the linear space of (31), and $\hat{\bar{y}}_{k-s}(sh \mid sh-h)$ is the projection of $\bar{y}_{k-s}(sh)$ onto the linear space of (32). Denote the corresponding innovation sequences as

$$\varepsilon_r(sh) = \bar{y}_r(sh) - \hat{\bar{y}}_r(sh \mid sh-h), \quad 0 \leqslant s < k-r, \tag{39}$$

$$\begin{aligned} \varepsilon_{k-s}(sh) &= \bar{y}_{k-s}(sh) - \hat{\bar{y}}_{k-s}(sh \mid sh-h), \\ & k-r \leqslant s \leqslant k, \end{aligned} \tag{40}$$

then we can say that $\left\{ \varepsilon_r(sh)|_{0 \leqslant s < k-r} \right\}$ and $\left\{ \varepsilon_{k-s}(sh)|_{k-r \leqslant s \leqslant k} \right\}$ are in fact mutually independent zero mean processes, which span the same linear space as in (31) and (32). Furthermore, we define

$$W_r(sh) \triangleq E\left[\varepsilon_r(sh)\varepsilon_r^T(sh) \right], \quad 0 \leqslant s < k-r, \tag{41}$$

$$W_{k-s}(sh) \triangleq E\left[\varepsilon_{k-s}(sh)\varepsilon_{k-s}^T(sh) \right], \quad k-r \leqslant s \leqslant k. \tag{42}$$

Define

$$\begin{aligned} \tilde{z}_r(sh \mid sh-h) &\triangleq z_r(sh) - \hat{z}_r(sh \mid sh-h), \\ & 0 \leqslant s < k-r \\ \tilde{z}_{k-s}(sh \mid sh-h) &\triangleq z_{k-s}(sh) - \hat{z}_{k-s}(sh \mid sh-h), \\ & k-r \leqslant s \leqslant k, \\ Z_{r,i}(sh) &\triangleq \mathrm{E}\left\{ z_{r,i}(sh)z_{r,i}^T(sh) \right\}, 0 \leqslant s < k-r, i \in \aleph_r, \\ Z_r(sh) &\triangleq \mathrm{E}\left\{ z_r(sh)z_r^T(sh) \right\} = diag[Z_{r,i}(sh)], 0 \leqslant s < k-r, \\ Z_{k-s,i}(sh) &\triangleq \mathrm{E}\left\{ z_{k-s,i}(sh)z_{k-s,i}^T(sh) \right\}, k-r \leqslant s \leqslant k, i \in \aleph_{k-s}, \\ Z_{k-s}(sh) &\triangleq \mathrm{E}\left\{ z_{k-s}(sh)z_{k-s}^T(sh) \right\} = diag[Z_{k-s,i}(sh)], k-r \leqslant s \leqslant k, \\ \widetilde{Z}_r(sh \mid sh-h) &\triangleq \mathrm{E}\left\{ \tilde{z}_r(sh \mid sh-h)\tilde{z}_r^T(sh \mid sh-h) \right\}, \\ & 0 \leqslant s < k-r, \\ \widetilde{Z}_{k-s}(sh \mid sh-h) & \\ &\triangleq \mathrm{E}\left\{ \tilde{z}_{k-s}(sh \mid sh-h)\tilde{z}_{k-s}^T(sh \mid sh-h) \right\}, \\ & k-r \leqslant s \leqslant k. \end{aligned}$$

The second-order moment matrices of the augmented states and the corresponding innovation sequences are

given below.

Lemma 3.1: The second-order moment matrices $Z_r(sh)$ and $Z_{k-s}(sh)$ are diagonal matrices, which can be calculated as follows.

For $0 \leqslant s < k - r$, the matrix $Z_r(sh)$ satisfies

$$Z_r(sh) = \text{diag}\left[Z_{r,j}(sh)\right], \tag{43}$$

$$Z_{r,j}(sh + h) = \sum_{i=1}^{N_r} p_{ij}^{(r)} A Z_{r,i}(sh) A^T + \sum_{i=1}^{N_r} \mu_{r,i}(sh) p_{ij}^{(r)} Q_w, \tag{44}$$

$$j \in \aleph_r,$$

where the initial value is

$$Z_{r,j}(0) = \bar{\Pi}_r(0) P(0), \quad j \in \aleph_r. \tag{45}$$

For $k - r \leqslant s \leqslant k$, the matrix $Z_{k-s}(sh)$ satisfies

$$Z_{k-s}(sh) = \text{diag}\left[Z_{k-s,j}(sh)\right], \tag{46}$$

$$Z_{k-s-1,j}(sh + h) = \sum_{i=1}^{N_{k-s}} p_{ij}^{(k-s)} A Z_{k-s,i}(sh) A^T + \sum_{i=1}^{N_{k-s}} \mu_{k-s,i}(sh) p_{ij}^{(k-s)} Q_w, \tag{47}$$

$$j \in \aleph_{k-s}.$$

Proof: See appendix A.

Denote

$$\mathcal{Z}_r(sh) \triangleq \left(Z_{r,1}(sh), \cdots, Z_{r,N_r}(sh)\right) \tag{48}$$

$$\mathcal{Z}_{k-s}(sh) \triangleq \left(Z_{k-s,1}(sh), \cdots, Z_{k-s,N_{k-s}}(sh)\right) \tag{49}$$

and define the operators

$$\mathcal{T}_r(sh, \mathcal{Z}_r) \triangleq (\mathcal{T}_{r,1}(sh, \mathcal{Z}_r), \cdots, \mathcal{T}_{r,N_r}(sh, \mathcal{Z}_r)), \tag{50}$$

$$\mathcal{T}_{k-s}(sh, \mathcal{Z}_{k-s}) \\ \triangleq (\mathcal{T}_{k-s,1}(sh, \mathcal{Z}_{k-s}), \cdots, \mathcal{T}_{k-s,N_{k-s}}(sh, \mathcal{Z}_{k-s})), \tag{51}$$

$$\mathcal{D}_r(sh) \triangleq \left(\mathcal{D}_{r,1}(sh), \cdots, \mathcal{D}_{r,N_r}(s)\right), \tag{52}$$

$$\mathcal{D}_{k-s}(sh) \triangleq \left(\mathcal{D}_{k-s,1}(sh), \cdots, \mathcal{D}_{k-s,N_{k-s}}(sh)\right) \tag{53}$$

by

$$\mathcal{T}_{r,j}(sh, \mathcal{Z}) \triangleq \sum_{i=1}^{N_r} p_{ij}^{(r)} A Z_{r,i}(sh) A^T, \quad 0 \leqslant s < k, \tag{54}$$

$$j = 1, \cdots, N_r,$$

$$\mathcal{T}_{k-s,j}(sh, \mathcal{Z}) \triangleq \sum_{i=1}^{N_{k-s}} p_{ij}^{(k-s)} A Z_{k-s,i}(sh) A^T, \tag{55}$$

$$k - r \leqslant s \leqslant k, j = 1, \cdots, N_{k-s},$$

$$\mathcal{D}_{r,j}(sh) \triangleq \sum_{i=1}^{N_r} \mu_{r,i}(sh) p_{ij}^{(r)} Q_w, \tag{56}$$

$$0 \leqslant s < k, j = 1, \cdots, N_r,$$

$$\mathcal{D}_{k-s,j}(sh) \triangleq \sum_{i=1}^{N_{k-s}} \mu_{k-s,i}(sh) p_{ij}^{(k-s)} Q_w,$$

$$k - r \leqslant s \leqslant k, j = 1, \cdots, N_{k-s}, \tag{57}$$

then the coupled Lyapunov equation can be rewritten as

$$\mathcal{Z}_r(sh+h) = \mathcal{T}_r(sh, \mathcal{Z}_r) + \mathcal{D}_r(sh), \quad 0 \leqslant s < k, \tag{58}$$

$$\mathcal{Z}_{k-s}(sh+h) = \mathcal{T}_{k-s}(sh, \mathcal{Z}_{k-s}) + \mathcal{D}_{k-s}(sh), \\ k - r \leqslant s \leqslant k, \tag{59}$$

Lemma 3.2: The second moment matrices $W_r(sh)$ ($0 \leqslant s < k - r$) and $W_{k-s}(sh)$ ($k - r \leqslant s \leqslant k$) satisfy the following equations:

$$W_r(sh) = \bar{G}_r \widetilde{Z}_r(sh \mid sh - h) \bar{G}_r^T + \bar{R}_r, \\ 0 \leqslant s < k - r, \tag{60}$$

$$W_{k-s}(sh) = \bar{G}_{k-s} \widetilde{Z}_{k-s}(sh \mid sh - h) \bar{G}_{k-s}^T + \bar{R}_{k-s}, \\ k - r \leqslant s \leqslant k. \tag{61}$$

Proof: See Appendix B.

Now, we are in the position to derive the main results of this paper. The covariance matrices of the estimation errors are given by the following theorem.

Theorem 3.1: The covariance matrices $\widetilde{Z}_r(sh+h \mid sh)$ ($0 \leqslant s < k - r$) and $\widetilde{Z}_{k-s-1}(sh+h \mid sh)$ ($k - r \leqslant s \leqslant k$) can be formulated as follows.

(i) For $0 \leqslant s < k - r$, $\widetilde{Z}_r(sh+h \mid sh)$ satisfies the following GCDRE:

$$\widetilde{Z}_r(sh+h \mid sh) = \bar{A}_r \widetilde{Z}_r(sh \mid sh - h) \bar{A}_r^T + \mathcal{E}(\mathcal{Z}_r(sh)) + Dg(\mathcal{D}_r(sh)) \\ - \bar{A}_r \widetilde{Z}_r(sh \mid sh - h) \bar{G}_r^T W_r(sh)^{-1} \bar{G}_r \widetilde{Z}_r(sh \mid sh - h) \bar{A}_r^T, \tag{62}$$

where the initial value is

$$\widetilde{Z}_r(0 \mid -h) = \text{diag} \left\{ \mu_1(0) P_0, \cdots, \mu_{N_r}(0) P_0 \right\} \tag{63}$$

and

$$\mathcal{E}(\mathcal{Z}_r(sh)) = \text{diag} \left\{ \sum_{i=1}^{N_r} p_{ij}^r A Z_{r,i}(sh) A^T \right\} - \bar{A}_r Z_r(sh) \bar{A}_r^T,$$

$$Dg(\mathcal{D}_r(sh)) = \text{diag} \left\{ \sum_{i=1}^{N_r} \mu_{r,i}(sh) p_{ij}^r Q_w \right\}.$$

(ii) For $k - r \leqslant s \leqslant k$, $\widetilde{Z}_{k-s-1}(sh+h \mid sh)$ satisfies the following GDCRE:

$$\widetilde{Z}_{k-s-1}(sh+h \mid sh) = \bar{A}_{k-s} \widetilde{Z}_{k-s}(sh \mid sh - h) \bar{A}_{k-s}^T + \mathcal{E}(\mathcal{Z}_{k-s}(sh)) \\ + Dg(\mathcal{D}_{k-s}(sh)) - \bar{A}_{k-s} \widetilde{Z}_{k-s}(sh \mid sh - h) \bar{G}_{k-s}^T \\ \times W_{k-s}(sh)^{-1} \bar{G}_{k-s} \widetilde{Z}_{k-s}(sh \mid sh - h) \bar{A}_{k-s}^T, \tag{64}$$

where

$$\bar{A}_{k-s} = \bar{\Pi}_{k-s}^T \otimes A,$$

$$\mathcal{E}(\mathcal{Z}_{k-s}(sh)) = \text{diag} \left\{ \sum_{i=1}^{N_{k-s}} p_{ij}^{(k-s)} A Z_{k-s,i}(sh) A^T \right\} \\ - \bar{A}_{k-s} Z_{k-s}(sh) \bar{A}_{k-s}^T,$$

$$Dg(\mathcal{D}_{k-s}(sh)) = \text{diag} \left\{ \sum_{i=1}^{N_{k-s}} p_{ij}^{(k-s)} \mu_{k-s,i}(sh) Q_w \right\}.$$

Proof: See Appendix C.

On the basis of Theorem 3.1, we will design the LMMSE estimator of $x(kh)$ by the innovation analysis method.

Theorem 3.2: Consider the multi-rate sampling system (1) and (2). The LMMSE $\hat{x}(kh \mid kh)$ is given by

$$\hat{x}(kh \mid kh) = \sum_{i=1}^{N_{k-s}} \hat{z}_{k-s,i}(kh \mid kh), \tag{65}$$

where $\hat{z}_{k-s,i}(kh \mid kh)$ is the i-th block element of $\hat{z}_{k-s}(kh \mid kh)$, which can be obtained iteratively by the following steps.

(i) For $0 \leqslant s < k - r$, $\hat{z}_r(sh \mid sh)$ can be calculated by the following difference equation with the initial value $\hat{z}_r(0 \mid -1) = E(z(0)) = 0$ and

$$\begin{aligned}
\hat{z}_r(sh \mid sh) =& \hat{z}_r(sh \mid sh - h) + \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T \\
& \times W_r(sh)^{-1} \left(\bar{y}_r(sh) - \hat{G}_r \hat{z}_r(sh \mid sh - h) \right),
\end{aligned} \tag{66}$$

$$\hat{z}_r(sh \mid sh - h) = \bar{A}_r \hat{z}_r(sh - h \mid sh - h). \tag{67}$$

(ii) For $k - r \leqslant s \leqslant k$, $\hat{z}_{k-s}(sh \mid sh)$ can be computed by the following recursive equation:

$$\begin{aligned}
\hat{z}_{k-s}(sh \mid sh) =& \hat{z}_{k-s}(sh \mid sh - h) + \widetilde{Z}_{k-s}(sh \mid sh - h)\bar{G}_{k-s}^T \\
& \times W_{k-s}(sh)^{-1} \left(\bar{y}_{k-s}(sh) - \hat{G}_{k-s}\hat{z}_{k-s}(sh \mid sh - h) \right).
\end{aligned} \tag{68}$$

$$\hat{z}_{k-s}(sh \mid sh - h) = \bar{A}_{k-s}\hat{z}_{k-s}(sh - h \mid sh - h). \tag{69}$$

Proof: On the ground of the relationship between $x(sh)$, $z_r(sh)$ and $z_{k-s}(sh)$, the proof follows easily from the derivation process of Theorem 3.1, and is therefore omitted here.

Note that Theorem 3.2 supply the state estimator at the measurement sampling time, but the estimator at the state update instant is not supplied.

Theorem 3.3: The original state estimator at the states update instant has the following form:

$$\hat{x}(t) = \hat{x}(kh \mid kh), \quad t = kh, \tag{70}$$

$$\hat{x}(t) = \hat{x}(kh + p \mid kh), \quad t = kh + p, \ p = 1, 2, \cdots, h, \tag{71}$$

where

$$\hat{x}(kh + p \mid kh) = A_o^p \hat{x}(kh \mid kh). \tag{72}$$

Proof: On the basis of the iteration of $x(kh)$ in (6) and the projection of $x(kh + p)$ on the linear space $\{y(0), y(h), \cdots, y(kh)\}$, we directly have the predictor of the state at instant kh.

4. Stationary Linear Filter

In this section, we will show that the GCDREs ($\widetilde{Z}_r(kh + h \mid kh)$ and $\widetilde{Z}_{k-s}(kh + h \mid kh)$) converge to a set of GCAREs when $k \to \infty$. Then, we develop a stationary filter according to the obtained GCAREs. First, some definitions and assumptions need to be made.

Definition 4.1 [30]**:** We say that system (1) is mean square stable (MSS) with $w(k) = 0$, if $E\left(\|x(k)\|^2\right) \to 0$ as $k \to \infty$ for any initial condition x_0 satisfying $E\left(\|x_0\|^2 < \infty\right)$.

Assumption 4.1: System (1) is MSS according to Definition 4.1.

Assumption 4.2: The Markov chain $\theta(kh)$ is ergodic.

Furthermore, we introduce the following GCAREs:

$$\begin{aligned}
\widetilde{Y}_r =& \bar{A}_r \widetilde{Y}_r \bar{A}_r^T + \mathcal{E}(\mathcal{Z}_r) + Dg(\mathcal{D}_r) \\
& - \bar{A}_r \widetilde{Y}_r \bar{G}_r^T (\bar{G}_r \widetilde{Y}_r \bar{G}_r^T + \bar{R}_r)^{-1} \bar{G}_r \widetilde{Y}_r \bar{A}_r^T,
\end{aligned} \tag{73}$$

$$\begin{aligned}
\widetilde{Y}_{k-s} =& \bar{A}_{k-s} \widetilde{Y}_{k-s} \bar{A}_{k-s}^T + \mathcal{E}(\mathcal{Z}_{k-s}) + Dg(\mathcal{D}_{k-s}) \\
& - \bar{A}_{k-s} \widetilde{Y}_{k-s} \bar{G}_{k-s}^T (\bar{G}_{k-s} \widetilde{Y}_{k-s} \bar{G}_{k-s}^T + \bar{R}_{k-s})^{-1} \\
& \times \bar{G}_{k-s} \widetilde{Y}_{k-s} \bar{A}_{k-s}^T.
\end{aligned} \tag{74}$$

Next, we will prove that GCDREs (62) and (64) converge to GCAREs (73) and (74), respectively, and there exists a set of positive semidefinite solutions to GCAREs (62) and (64). It can be seen that if the GCDRE (62) for $\widetilde{Z}_r(sh+h\mid sh)$ $(0\leqslant s< k-r)$ converges, then after finite iterations, (64) converges to (74) as well. In addition, if there is a stable solution to (73), there is also a stable solution to (74). Therefore, it is only necessary for us to prove the convergence of (62) and find the existence condition for the stable solution to (73).

Based on the Proposition 3.36 in [30], and in light of Assumptions 4.1 and 4.2, we have $\mathcal{Z}_r(sh)\to\mathcal{Z}_r$ as $s\to\infty$, where $\mathcal{Z}_r=\left(Z_{r,1},\cdots,Z_{r,N_r}\right)$ is the unique solution to the following coupled algebraic Lyapunov equation:

$$Z_{r,j}=\sum_{i=1}^{N_r}p_{ij}^r\left(AZ_{r,i}A^T+\mu_iQ_w\right),j\in\mathbb{N}_r.\tag{75}$$

In what follows, for any matrix $Z_r\in\mathbb{B}\left(\mathbb{R}^{Nn}\right),Z_r\geqslant0$, define $\mathrm{T}(Z_r)\in\mathbb{B}\left(\mathbb{R}^p,\mathbb{R}^{Nn}\right)$ as

$$\mathrm{T}(Z_r)\triangleq-\bar{A}_rZ_r\bar{G}_r^T(\bar{G}_rZ_r\bar{G}_r^T+\bar{R}_r)^{-1}.\tag{76}$$

Recalling from (73), we make the following definition:

$$\begin{aligned}\mathrm{T}_r(sh)\triangleq&-\bar{A}_r\widetilde{Z}_r(sh\mid sh-h)\bar{G}_r^T\\&\times\left(\bar{G}_r\widetilde{Z}_r(sh\mid sh-h)\bar{G}_r^T+\bar{R}_r\right)^{-1}.\end{aligned}\tag{77}$$

Substituting (77) into (62) yields the following form:

$$\begin{aligned}\widetilde{Z}_r(sh+h\mid sh)=&(\bar{A}_r+\mathrm{T}_r(sh)\bar{G}_r)\widetilde{Z}_r(sh\mid sh-h)(\bar{A}_r+\mathrm{T}_r(sh)\bar{G}_r)^T\\&+\mathcal{E}(\mathcal{Z}_r(sh))+Dg(\mathcal{D}_r(sh))+\mathrm{T}_r(sh)\bar{R}_r\mathrm{T}_r(sh)^T.\end{aligned}\tag{78}$$

Notice from Lemma 3.2 that the inverse matrix of $W_r(sh)$ is well defined since

$$\bar{G}_r\widetilde{Z}_r(sh\mid sh-h)\bar{G}_r^T+\bar{R}_r\geqslant\bar{R}_r>0.\tag{79}$$

The following is the main result of this section where the core task is to establish the asymptotic convergence of $\widetilde{Z}_r(sh\mid sh-h)$. Before that, we need to present two auxiliary results.

Let κ be given such that $\inf_{\ell\geqslant\kappa}\mu_{r,i}(\ell h)>0$ holds for all $i\in\mathbb{N}_r$. Define

$$\alpha_{r,i}(sh)\triangleq\inf_{\ell\geqslant\kappa}\mu_{r,i}(\ell h+\kappa h).$$

Obviously, for $i\in\mathbb{N}_r$, we have

$$\mu_{r,i}(sh+\kappa h)\geqslant\alpha_{r,i}(sh)\geqslant\alpha_{r,i}(sh-h).\tag{80}$$

Define $\bar{\mathcal{Z}}_r(sh)\triangleq\left(\bar{\mathcal{Z}}_{r,1}(sh),\cdots,\bar{\mathcal{Z}}_{r,N_r}(sh)\right)$ with $\bar{\mathcal{Z}}_{r,j}(0)=0,j\in\mathbb{N}_r$ and

$$\bar{\mathcal{Z}}_{r,j}(sh+h)=\sum_{i=1}^N p_{ij}\left(A\bar{\mathcal{Z}}_{r,i}(sh)A^T+\alpha_{r,i}(sh)Q_w\right).\tag{81}$$

Recalling the definition of $\mathcal{Z}_r=(Z_{r,1},...,Z_{r,N_r})$ in (75), we obtain the following result.

Lemma 4.1: $\bar{\mathcal{Z}}_r(sh)\overset{s\to\infty}{\to}\mathcal{Z}_r$ and for each $k=0,1,2,\cdots,0\leqslant s< k-r$, we have

$$\mathcal{Z}_r(sh+\kappa h)\geqslant\bar{\mathcal{Z}}_r(sh)\geqslant\bar{\mathcal{Z}}_r(sh-h).\tag{82}$$

Proof: See Appendix D.

Define

$$\begin{aligned}V_r(sh+h)=&\bar{A}_rV_r(sh)\bar{A}_r^T+\mathcal{E}(\bar{\mathcal{Z}}_r(sh))+\mathrm{diag}\left[\sum_{i=1}^{N_r}\alpha_{r,i}(sh)p_{ij}Q_w\right]\\&+\bar{A}_rV_r(sh)\bar{G}_r^T\left(\bar{G}_rV_r(sh)\bar{G}_r^T+\bar{R}_r\right)^{-1}\bar{G}_rV_r(sh)\bar{A}_r^T\end{aligned}\tag{83}$$

where $V(0)=0$. Notice that from Lemma 3.2, the inverse matrix of $\bar{G}_rV_r(sh)\bar{G}_r^T+\bar{R}_r$ is well defined.

Lemma 4.2: For each $k=0,1,\cdots,0\leqslant s< k-r$, we have

$$0\leqslant V_r(sh)\leqslant V_r(sh+h)\leqslant\widetilde{Z}_r(sh+h+\kappa h\mid sh+\kappa h).\tag{84}$$

Proof: See Appendix E.

In the following theorem, we first demonstrate the existence of a unique positive semi-definite solution to Y_r for the GCARE. Then, we prove that for a positive integer $\kappa > 0$, there exist lower and upper bounds $V(sh)$ and $Y_r(sh + kh)$ on $\widetilde{Z}_r(sh + \kappa h \mid sh + \kappa h - h)$ such that $\widetilde{Z}_r(sh + \kappa h \mid sh + \kappa h - h)$ asymptotically converges to Y_r.

Theorem 4.1: Suppose that the Markov chain $\{\theta_r(sh)\}$ is ergodic and system (1)-(2) is MSS. For any $\mathcal{Z}_r(0) = \left(\mathcal{Z}_{r,1}(0), \cdots, \mathcal{Z}_{r,N_r}(0) \right)$ with $\mathcal{Z}_{r,i}(0) \geqslant 0, i \in \mathbb{N}_r$ and $\widetilde{Z}_r(0 \mid -h) = \mathrm{diag}\left[\mathcal{Z}_{r,i}(0) \right] - \hat{Z}_r(0 \mid -h) \geqslant 0$, $\widetilde{Z}_r(kh + h \mid kh)$ given by (58) and (62) satisfies

$$\widetilde{Z}_r(sh + h \mid sh) \stackrel{s \to \infty}{\to} Y_r$$

where Y_r is the unique positive semidefinite solution to the following GARE:

$$X = \bar{A}_r X \bar{A}_r^T + Q_w + \mathcal{E}(\mathcal{Z}_r) - \bar{A}_r X \bar{G}_r^T (\bar{G}_r X \bar{G}_r^T + \bar{R}_r)^{-1} \bar{G}_r X \bar{A}_r^T. \tag{85}$$

Moreover,

$$r_\sigma(\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r) < 1.$$

Proof: See Appendix F.

Corollary 4.1: The stationary LMMSE filter $\hat{x}(kh \mid kh)$ is

$$\hat{x}^{(s)}(kh \mid kh) = \sum_{i=1}^{N_{k-s}} \hat{z}_{k-s,i}^{(s)}(kh \mid kh), \tag{86}$$

where $\hat{z}_{k-s,i}^{(s)}(kh \mid kh)$ is the i-th block element of $\hat{z}_{k-s}^{(s)}(kh \mid kh)$ which can be obtained iteratively by the following steps.

(i) For $0 \leqslant s < k - r$, $\hat{z}_r^{(s)}(sh \mid sh)$ can be calculated by the following difference equation:

$$\begin{aligned} \hat{z}_r^{(s)}(sh \mid sh) =& \hat{z}_r(sh \mid sh - h) + Y_r \bar{G}_r^T (\bar{G}_r Y_r \bar{G}_r^T + \bar{R}_r)^{-1} \\ &\times \left(\bar{y}_r(sh) - A\hat{z}_r^{(s)}(sh \mid sh - h) \right), \\ & 0 \leqslant s < k - r, \end{aligned} \tag{87}$$

$$\hat{z}_r^{(s)}(sh \mid sh - h) = \bar{A}_r \hat{z}_r^{(s)}(sh - h \mid sh - h), \tag{88}$$

(ii) For $k - r \leqslant s \leqslant k$, $\hat{z}_{k-s}^{(s)}(sh \mid sh)$ can be calculated as follows:

$$\begin{aligned} \hat{z}_{k-s}^{(s)}(sh \mid sh) =& \hat{z}_{k-s}^{(s)}(sh \mid sh - h) + Y_{k-s} \bar{G}_{k-s}^T (\bar{G}_{k-s} Y_{k-s} \bar{G}_{k-s}^T + \bar{R}_{k-s})^{-1} \\ &\times \left(\bar{y}_{k-s}(sh) - A\hat{z}_{k-s}^{(s)}(sh \mid sh - h) \right), \end{aligned} \tag{89}$$
$$\hat{z}_{k-s}^{(s)}(sh \mid sh - h) = \bar{A}_{k-s} \hat{z}_{k-s}^{(s)}(sh - h \mid sh - h).$$

Proof: From Theorem 4.2, it can be obtained that the error covariance of the infinite time state estimator converges to a unique solution. Therefore, the result can be obtained immediately by replacing the error covariances in (66) and (69) with the stable solution in the Theorem 4.2.

Corollary 4.2: The infinite time estimator at the state update instant has the following form:

$$\hat{x}^{(s)}(t) = \hat{x}^{(s)}(kh \mid kh), \ t = kh, \tag{90}$$

$$\hat{x}^{(s)}(t) = \hat{x}^{(s)}(kh + p \mid kh), \ t = kh + p, \ p = 1, 2, \cdots, h, \tag{91}$$

where

$$\hat{x}^{(s)}(kh + p \mid kh) = A_o^p \hat{x}^{(s)}(kh \mid kh). \tag{92}$$

Proof: For $k \to \infty$, we have formulas (90) and (91) by calculating the projection of formula (92) on linear space $\{y(0), y(h), \cdots, y(kh)\}$. This completes the proof.

5. Numerical Example

Consider the following linear discrete-time system:

$$x(kh + h) = \begin{bmatrix} 2 & 1.1 \\ -1.7 & -0.8 \end{bmatrix} x(kh) + \begin{bmatrix} 2 \\ 0.1 \end{bmatrix} w(kh),$$

$$y_0(kh) = \gamma_0(kh) \begin{bmatrix} 4 & 2 \end{bmatrix} x(kh) + v_0(kh),$$

$$y_1(kh) = \gamma_1(kh) \begin{bmatrix} 3 & 1 \end{bmatrix} x(kh - h) + v_1(kh),$$

where $w(kh)$ and $v_i(kh)$ $(i = 0, 1)$ are white noises with zero means and covariance matrices $Q = 1$ and $R_i = 1, i = 0, 1$, respectively. Set the initial value x_0 and its covariance matrix to be

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad P_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let the channel packet loss parameters be

$$p_1 = 0.80, \quad p_2 = 0.20,$$
$$q_1 = 0.75, \quad q_2 = 0.25.$$

For the proposed optimal state estimator, 50 Monte Carlo experiments are carried out and the following simulation results are obtained. Figures 1 and 2 show the path of the Markov packet loss in two periods. Figure 3 shows the trajectory of the real and its estimated value of the first component of $x(k)$. Figure 4 shows the trajectory of the real state and the estimated value of the second component of $x(k)$. From the simulation results, it can be seen that the filter derived in this paper has good tracking performances.
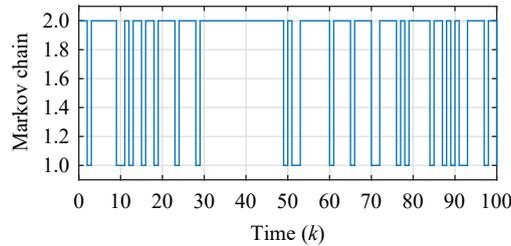
Figure 1. An sample path of packet loss of the first channel.
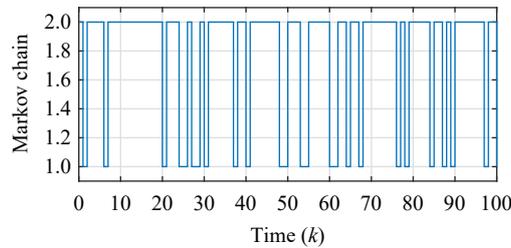
Figure 2. An sample path of packet loss of the second channel.
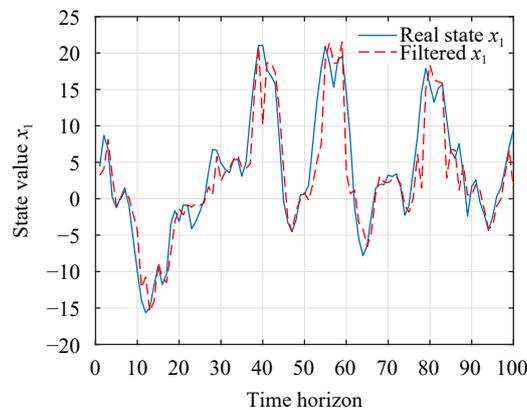
Figure 3. Simulation of the state value (blue line) and estimation (red line) of the first state component trajectory x_1.
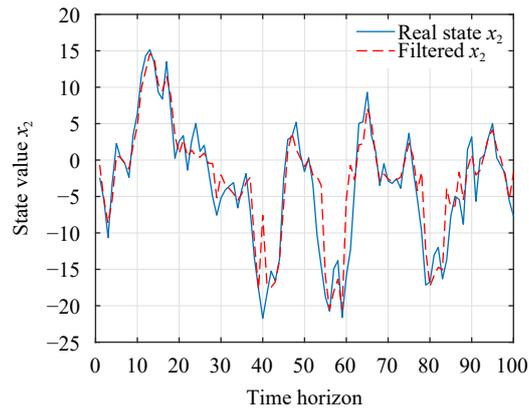
Figure 4. Simulation of the state value (blue line) and estimation (red line) of the first state component trajectory x_2.

6. Conclusion

This paper has mainly studied the linear state estimation problem of multi-rate sampling systems with erasure channels. For the multi-channel packet loss with unknown Markov jump parameters, a new Markov chain has been established to simulate the random packet loss process. By introducing some new state variables and using the reorganization technology, the original system model has been transformed into a delay-free Markov jump system. Then, a set of GDREs has been conducted based on the Lyapunov equations with analytical solutions obtained. Finally, convergence analysis has been conducted and a stationary filter has been obtained.

Author Contributions: Zhenglu Sun: Writing, reviewing and editing; software and simulation. **Chunyan Han:** Investigation; methodology; reviewing and editing; project administration.

Data Availability Statement: This article does not include any new data, as no data were created or analyzed in this study.

Conflicts of Interest: The authors declare no conflict of interest.

Appendix A. Proof of Lemma 3.1

Proof: From (31), it can be shown that the following Lyapunov-like recurrent equation for the second moment matrix $Z_{r,i}(sh)$ holds where $0 \leqslant s \leqslant k, i = 1, \cdots, N_r$.

$$
\begin{aligned}
Z_{r,j}(sh + h) &= \mathrm{E}\left\{ z_{r,i}(sh) z_{r,i}^T(sh) \right\} \\
&= \sum_{i=1}^{N_r} p_{ij}^{(r)} A Z_{r,i}(sh) A^T + \sum_{i=1}^{N_r} \mu_{r,i}(sh) p_{ij}^{(r)} Q_w, \\
&\quad j = 1, \cdots, N_r.
\end{aligned}
\tag{93}
$$

Here, the initial value is $Z_{r,i}(0) = \mu_{r,i}(0) P_0$ for $i = 1, \cdots, N$.

Similarly, (47) can be obtained. The proof is completed.

Appendix B. Proof of Lemma 3.2

Proof: In view of (35)-(36) and the projection theorem, it can be obtained that

$$
\hat{y}_r(sh \mid sh - h) = \bar{G}_r \hat{z}_r(sh \mid sh - h), \ 0 \leqslant s < k - r,
\tag{94}
$$

$$\hat{\bar{y}}_{k-s}(sh \mid sh - h) = \bar{G}_r \hat{z}_{k-s}(sh \mid sh - h), \ k - r \leqslant s \leqslant k. \tag{95}$$

Then, it can be deduced from (39) and (40) that

$$\varepsilon_r(sh) = \bar{G}_r \tilde{z}_r(sh \mid sh - h) + \bar{v}_r(sh), \tag{96}$$

$$\varepsilon_{k-s}(sh) = \bar{G}_{k-s} \tilde{z}_{k-s}(sh \mid sh - h) + \bar{v}_{k-s}(sh). \tag{97}$$

By the definitions of $W_r(sh)$ and $W_{k-s}(sh)$, we have

$$\begin{aligned} W_r(sh) &= E\left[\varepsilon_r(sh)\varepsilon_r^T(sh)\right] \\ &= \bar{G}_r \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T + \bar{R}_r, \quad 0 \leqslant s < k - r, \end{aligned} \tag{98}$$

$$\begin{aligned} W_{k-s}(sh) &= E\left[\varepsilon_{k-s}(sh)\varepsilon_{k-s}^T(sh)\right] \\ &= \bar{G}_{k-s} \widetilde{Z}_{k-s}(sh \mid sh - h)\bar{G}_{k-s}^T + \bar{R}_{k-s}, \quad k - r \leqslant s \leqslant k. \end{aligned} \tag{99}$$

This completes the proof.

Appendix C. Proof of Theorem 3.1

Proof: We first consider the case of $0 \leqslant s < k - r$. Define $\hat{z}_r(sh \mid sh)$ as the projection of $z_r(sh)$ onto the linear space $\{\bar{y}_r(0), \cdots, \bar{y}_r(s)\}$. Based on the projection formula, we have

$$\begin{aligned} \hat{z}_r(sh \mid sh) &= \hat{z}_r(sh \mid sh - h) + E\left[z_r(sh)\varepsilon_r^T(sh)\right] \\ &\quad \times E\left[\varepsilon_r(sh)\varepsilon_r^T(sh)\right]^{-1} \varepsilon_r(sh). \end{aligned} \tag{100}$$

From (96), we have

$$E\left[z_r(sh){\varepsilon_r}^T(sh)\right] = \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T. \tag{101}$$

Substituting (101) into (100), we have

$$\begin{aligned} \hat{z}_r(sh \mid sh) &= \hat{z}_r(sh \mid sh - h) + \widetilde{Z}_r(sh \mid sh - h){\bar{G}_r}^T \\ &\quad \times W_r(sh)^{-1}\left(\bar{y}_r(sh) - \hat{\bar{y}}_r(sh \mid sh - h)\right). \end{aligned} \tag{102}$$

On the other hand, it can be seen from (32) that

$$\begin{aligned} \hat{z}_r(sh + h \mid sh) &= \bar{A}_r \hat{z}_r(sh \mid sh) \\ &\quad + E\left[\mathcal{B}_{r,1}(sh) \mid \varepsilon_r(0), \varepsilon_r(h), \cdots, \varepsilon_r(sh)\right] \\ &\quad + E\left[\mathcal{B}_{r,2}(sh) \mid \varepsilon_r(0), \varepsilon_r(h), \cdots, \varepsilon_r(sh)\right]. \end{aligned} \tag{103}$$

From Assumptions 1-2, we have

$$\begin{aligned} \hat{z}_r(sh + h \mid sh) &= \bar{A}_r \hat{z}_r(sh \mid sh). \\ &= \bar{A}_r \hat{z}_r(sh \mid sh - h) + F_r(sh)\left(\bar{y}_r(sh) - \hat{\bar{y}}_r(sh \mid sh - h)\right) \end{aligned} \tag{104}$$

with

$$F_r(sh) = \bar{A}_r \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T W_r(sh)^{-1}.$$

In view of (32), (102) and (104), we obtain

$$\begin{aligned} \tilde{z}_r(sh + h \mid sh) &= z_r(sh + h) - \hat{z}_r(sh + h \mid sh) \\ &= \bar{A}_r z_r(sh) + \mathcal{B}_{r,1}(sh) + \mathcal{B}_{r,2}(sh) \\ &\quad - \bar{A}_r(\hat{z}_r(sh \mid sh - h) + \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T W_r(sh)^{-1} \\ &\quad \times (\bar{y}_r(sh) - \hat{\bar{y}}_r(sh \mid sh - h))) \\ &= \bar{A}_r \tilde{z}_r(sh \mid sh - h) + \mathcal{B}_{r,1}(sh) + \mathcal{B}_{r,2}(sh) \\ &\quad - F_r(sh)\bar{G}_r \tilde{z}_r(sh \mid sh - h) - F_r(sh)v_r(sh). \end{aligned} \tag{105}$$

Setting

$$\mathcal{B}_{r,0}(sh) = (\bar{A}_r - F_r(sh)\bar{G}_r)\tilde{z}_r(sh \mid sh - h), \mathcal{B}_{r,3}(sh) = F_r(sh)v_r(sh), \tag{106}$$

we have

$$\tilde{z}_r(sh + h \mid sh) = \mathcal{B}_{r,0}(sh) + \mathcal{B}_{r,1}(sh) + \mathcal{B}_{r,2}(sh) + \mathcal{B}_{r,3}(sh). \tag{107}$$

On the ground of the assumptions given in Section 2, we have

$$E\{\mathcal{B}_{r,0}(sh)\mathcal{B}_{r,l}^T(sh)\} = 0, \; l = 1, 2, 3 \tag{108}$$

$$E\{\mathcal{B}_{r,1}(sh)\mathcal{B}_{r,l}^T(sh)\} = 0, \; l = 2, 3 \tag{109}$$

$$E\{\mathcal{B}_{r,2}(sh)\mathcal{B}_{r,3}^T(sh)\} = 0. \tag{110}$$

Furthermore, the covariance matrices of $\mathcal{B}_{r,l}(sh)$, $l = 0, 1, \cdots, 3$, can be derived as follows:

$$E\{\mathcal{B}_{r,0}(sh)\mathcal{B}_{r,0}^T(sh)\} = (\bar{A}_r - F_r(sh)\bar{G}_r)\widetilde{Z}_r(sh \mid sh - h)(\bar{A}_r - F_r(sh)\bar{G}_r)^T, \tag{111}$$

$$E\{\mathcal{B}_{r,1}(sh)\mathcal{B}_{r,1}^T(sh)\} = \text{diag}\Big\{\sum_{i=1}^{N_r} p_{ij}^r A Z_{r,i}(sh) A^T\Big\} - \bar{A}_r Z_r(sh)\bar{A}_r^T, \tag{112}$$

$$E\{\mathcal{B}_{r,2}(sh)\mathcal{B}_{r,2}^T(sh)\} = \text{diag}\Big\{\sum_{i=1}^{N_r} \mu_{r,i}(sh) p_{ij}^r Q_w\Big\}. \tag{113}$$

$$E\{\mathcal{B}_{r,3}(sh)\mathcal{B}_{r,3}^T(sh)\} = F_r(sh)R_r(sh)F_r^T(sh). \tag{114}$$

In light of (108)–(114), the GCDRE can be obtained as follows:

$$\begin{aligned}
\widetilde{Z}_r(sh + h \mid sh) =& \bar{A}_r \widetilde{Z}_r(sh \mid sh - h)\bar{A}_r^T + \mathcal{E}(\mathcal{Z}_r(sh)) + Dg(\mathcal{D}_r(sh)) \\
&- \bar{A}_r \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T(\bar{G}_r \widetilde{Z}_r(sh \mid sh - h)\bar{G}_r^T + \bar{R}_r)^{-1} \\
&\times \bar{G}_r \widetilde{Z}_r(sh \mid sh - h)\bar{A}_r^T.
\end{aligned} \tag{115}$$

(64) follows immediately from the similar derivation line of $\widetilde{Z}_r(sh \mid sh - h)$. This completes the proof.

Appendix D. Proof of Lemma 4.1

Proof: Let us now show (82) by the mathematical induction method. For $sh = 0$, the result is obvious, since $\mathcal{Z}_r(\kappa h) \geqslant 0 = \bar{\mathcal{Z}}_r(0)$ and $\bar{\mathcal{Z}}_r(h) \geqslant 0 = \bar{\mathcal{Z}}_r(0)$. Suppose that (82) holds for sh. Then, from (80) and (82), we have

$$\begin{aligned}
\mathcal{Z}_{r,j}(sh + h + \kappa h) &= \sum_{i=1}^{N_r} p_{ij}\left(A\mathcal{Z}_{r,i}(sh + \kappa h)A^T + \mu_{r,i}(sh + \kappa h)Q_w\right) \\
&\geqslant \sum_{i=1}^{N_r} p_{ij}\left(A\bar{\mathcal{Z}}_{r,i}(sh)A^T + \alpha_{r,i}(sh)Q_w\right) \\
&= \bar{\mathcal{Z}}_{r,j}(sh + h) \\
&\geqslant \sum_{i=1}^{N_r} p_{ij}\left(A\bar{\mathcal{Z}}_{r,i}(sh - h)A^T + \alpha_{r,i}(sh - h)Q_w\right) \\
&= \bar{\mathcal{Z}}_{r,j}(sh).
\end{aligned} \tag{116}$$

(82) is shown by the mathematical induction method.

Appendix E. Proof of Lemma 4.2

Proof: Let us show (84) by the mathematical induction method. Set

$$S_r(sh) = -\bar{A}V_r(sh)\bar{G}_r^T\left(\bar{G}_r V_r(sh)\bar{G}_r^T + \bar{R}_r\right)^{-1}.$$

If $V_r(sh) \leqslant \widetilde{Z}_r(sh + \kappa h \mid sh - h + \kappa h)$, it follows from (80) and (82) that

$$
\begin{aligned}
V_r(sh+h) =& (\bar{A}_r + \mathrm{T}_r(sh+\kappa h)\bar{G}_r)V_r(sh)(\bar{A}_r + \mathrm{T}_r(sh+\kappa h)\bar{G}_r^T) \\
& + \mathcal{E}(\bar{\mathcal{Z}}_r(sh)) + \mathrm{diag}\left[\sum_{i=1}^{N_r} \alpha_{r,i}(sh)p_{ij}Q_w\right] \\
& + \mathrm{T}_r(sh+\kappa h)\bar{R}_r\,\mathrm{T}(sh+\kappa h)^T - (\mathrm{T}_r(sh+\kappa h) - S_r(sh)) \\
& \times \left(\bar{G}_r V(sh)\bar{G}_r^T + \bar{R}_r\right)(\mathrm{T}_r(sh+\kappa h) - S_r(sh))^T \\
\leqslant& (\bar{A}_r + \mathrm{T}_r(sh+h)\bar{G}_r)\widetilde{Z}_r(sh+\kappa h \mid sh-h+\kappa h) \\
& \times (\bar{A}_r + \mathrm{T}_r(sh+\kappa h)\bar{G}_r)^T + \mathcal{E}(\mathcal{Z}_r(sh+\kappa h)) \\
& + \mathrm{diag}\left[\sum_{i=1}^{N_r} \mu_{r,i}(sh+\kappa h)p_{ij}Q_w\right] + \mathrm{T}_r(sh+\kappa h)\bar{R}_r\,\mathrm{T}_r(sh+\kappa h)^T \\
=& \widetilde{Z}_r(sh+h+\kappa h \mid sh+\kappa h).
\end{aligned}
\tag{117}
$$

Obviously, $V(0) = 0 \leqslant \widetilde{Z}_r(\kappa h \mid \kappa h - h)$, showing that $V_r(sh) \leqslant \widetilde{Z}_r(sh+\kappa h \mid sh+h+\kappa h)$ holds for all $k = 0, 1, 2, \cdots, 0 \leqslant s < k - r$. Similarly, if $V_r(sh) \geqslant V_r(sh-h)$, from (80) and (82), we have

$$
\begin{aligned}
V_r(sh) =& (\bar{A} + S_r(sh)\bar{G}_r)V_r(sh-h)(\bar{A} + S_r(sh)\bar{G}_r)^T \\
& + \mathcal{E}(\mathcal{Z}_r(sh-h)) + \mathrm{diag}\left[\sum_{i=1}^{N_r} \alpha_{r,i}(sh-h)p_{ij}^r Q_w\right] \\
& + S_r(sh)\bar{R}_r S_r(sh)^T - (S_r(sh) - S_r(sh-h)) \\
& \times \left(\bar{G}_r V_r(sh-h)\bar{G}_r^T + \bar{R}_r\right)^{-1}(S_r(sh) - S_r(sh-h))^T \\
\leqslant& (\bar{A}_r + S_r(sh)\bar{G}_r)V_r(sh)(\bar{A}_r + S_r(sh)\bar{G}_r)^T + \mathcal{E}(\mathcal{Z}_r(sh)) \\
& + \mathrm{diag}\left[\sum_{i=1}^{N_r} \alpha_{r,i}(sh)p_{ij}^r Q_w\right] + S_r(sh)\bar{R}_r S_r(sh)^T \\
=& V_r(sh+h).
\end{aligned}
\tag{118}
$$

Since $V_r(0) = 0 \leqslant V_r(h)$, the induction argument is completed for (84).

Appendix F. Proof of Theorem 4.1

Proof: From the MSS of system (1)-(2), we have $r_\sigma(A) < 1$ from Proposition 3.6 in [30]. Based on the standard results for GAREs, we conclude that there exists a unique positive semi-definite solution $Y_r \in \mathbb{B}\left(\mathbb{R}^{Nn}\right)$ to (85) where $r_\sigma(\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r) < 1$ (see [48]).

Furthermore, Y_r satisfies

$$
\begin{aligned}
Y_r =& (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)Y(\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)^T + \mathcal{E}(\mathcal{Z}_r) \\
& + Q_w + \mathrm{T}(Y_r)\bar{V}_r\mathrm{T}(Y_r)^T.
\end{aligned}
\tag{119}
$$

Define $Y(0) = \widetilde{Z}(0 \mid -1)$ and

$$
\begin{aligned}
Y_r(sh+h) =& (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)Y_r(sh)(\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)^T \\
& + \mathcal{E}(\mathcal{Z}_r(sh)) + Q_w + \mathrm{T}(Y_r)\bar{R}_r\mathrm{T}(Y_r)^T.
\end{aligned}
\tag{120}
$$

Let us show by induction that $Y_r(sh) \geqslant \widetilde{Z}_r(sh \mid sh-h)$. Since

$$
\begin{aligned}
\widetilde{Z}_r(sh+h \mid sh) =& \mathcal{E}(\mathcal{Z}_r(sh)) + Q_w + (\bar{A} + \mathrm{T}(Y)\bar{G}_r)\widetilde{Z}_r(sh \mid sh-h) \\
& \times (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)^T + \mathrm{T}(Y_r)Q_w\mathrm{T}(Y_r)^T \\
& - (\mathrm{T}_r(sh) - \mathrm{T}(Y_r))\left(\bar{G}_r\widetilde{Z}_r(sh \mid sh-h)\bar{G}_r^T + \bar{R}_r\right) \\
& \times (\mathrm{T}_r(sh) - \mathrm{T}(Y_r))^T,
\end{aligned}
\tag{121}
$$

from (120) and (121), we have

$$
\begin{aligned}
&Y_r(sh+h) - \widetilde{Z}_r(sh+h \mid sh) \\
&= (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)(Y_r(sh) - \widetilde{Z}_r(sh \mid sh-h)) \\
&\times (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)^T + (\mathrm{T}_r(sh) - \mathrm{T}(Y_r)) \\
&\times (\bar{G}_r\widetilde{Z}_r(sh \mid sh-h)\bar{G}_r^T + \bar{R}_r t)(\mathrm{T}_r(sh) - \mathrm{T}(Y_r))^T.
\end{aligned}
\tag{122}
$$

By definition, we have $Y_r(0) = \widetilde{Z}_r(0 \mid -1)$. Suppose $Y_r(sh) \geqslant \widetilde{Z}_r(sh \mid sh-h)$. From (122), we have $Y_r(sh+h) \geqslant \widetilde{Z}_r(sh+h \mid h)$. Therefore, we have shown by induction that $Y_r(sh) \geqslant \widetilde{Z}_r(sh \mid sh-h)$ holds for all k.

From the MSS and ergodicity of the Markov chain, we obtain $\mathcal{Z}_r(sh) \overset{k\to\infty}{\to} \mathcal{Z}_r$. Based on Proposition 3.36 in [30], it can be concluded that $Y_r(sh) \overset{k\to\infty}{\to} \bar{Y}_r$, where $\bar{Y}$ satisfies

$$
\begin{aligned}
\bar{Y}_r =& (\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)\bar{Y}_r(\bar{A}_r + \mathrm{T}(Y_r)\bar{G}_r)^T \\
&+ \mathcal{E}(\mathcal{Z}_r) + Q_w + \mathrm{T}(Y_r)\bar{R}_r\,\mathrm{T}(Y_r)^T
\end{aligned}
\tag{123}
$$

and $\bar{Y}_r$ is the unique solution to (123). Recalling that Y_r satisfies (119), we know that Y_r is also a solution to (123). From the uniqueness of the solution, we have $\bar{Y}_r = Y_r$. Therefore, we have

$$
\widetilde{Z}_r(sh \mid sh-h) \leqslant Y_r(sh)
\tag{124}
$$

and $Y_r(sh) \overset{k\to\infty}{\to} Y_r$.

It follows from (124) and (84) that $0 \leqslant V_r(sh) \leqslant Y_r(sh+h) \leqslant Y_r(sh+h+\kappa h)$. Thus, we can conclude that $V_r(sh) \uparrow V_r$ when $k \to \infty$ for $V_r \geqslant 0$. Moreover, from the fact that $\alpha_{r,i}(sh) \overset{k\to\infty}{\to} \mu_{r,i}$ and $\mathcal{Z}_r(sh) \overset{k\to\infty}{\to} \mathcal{Z}_r$, we have V_r satisfies (85). From the uniqueness of the positive semi-definite solution to (85), we conclude that $V_r = Y_r$. From (124) and (84), we have $V_r(sh) \leqslant \widetilde{Z}_r(sh+\kappa h \mid sh-h+\kappa h) \leqslant Y_r(sh+\kappa h)$. Since $V_r(sh) \to Y_r$ and $Y_r(sh) \to Y_r$ as $k \to \infty$, we have $\widetilde{Z}_r(sh \mid sh-h) \overset{k\to\infty}{\to} Y_r$. This completes the proof.

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